Chapter 11

Bisection method

$f(a_i)$ and $f(b_i)$ must have opposite signs then

i.e., $f(a_i).f(b_i)<0$ then $c_i=\frac{a_i+b_i}{2}$

First iteration $\quad f(a_0).f(b_0)<0$

$c_o=\frac{a_o+b_o}{2}$ then find $f(c_o)$

Check

Either	Or
$f(a_0).f(c_0)<0$	$f(c_0).f(b_0)<0$
Update	
$a_o=a_1$ and $c_0=b_1$	$c_o=a_1$ and $b_o=b_1$

Continuing in this manner

Regula-Falsi Mehtod

$f(a_i)$ and $f(b_i)$ must have opposite signs then

i.e., $f(a_i).f(b_i)<0$

$$c_i=a_i-\frac{(a_i-b_i)f(a_i)}{f(a_i)-f(b_i)}$$

First iteration $\quad f(a_0).f(b_0)<0$

$c_o=a_o-\frac{(a_o-b_o)f(a_o)}{f(a_o)-f(b_o)}$ then find $f(c_o)$

Check

Either	Or
$f(a_0).f(c_0)<0$	$f(c_0).f(b_0)<0$
Update	
$a_o=a_1$ and $c_0=b_1$	$c_o=a_1$ and $b_o=b_1$

Continuing in this manner

Newton Raphson's Method

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)} \qquad i=0,1,2,3,\ldots$$

Take $i=0$ and x_o is given

$$x_1=x_{0+1}=x_0-\frac{f(x_0)}{f'(x_0)}$$

Now take $i=1$

$$x_2=x_{1+1}=x_1-\frac{f(x_1)}{f'(x_1)}$$

Continuing in this manner

Exercise 11.1

Q1. Find an interval $a\le x\le b$ at which $f(a)$ and $f(b)$ have opposite signs for following functions:

a). $f(x)=e^x-2-x$

Sol: Given $f(x)=e^x-2-x$..................(1)

Put $x=0$ in equation (1) we get

$f(0)=e^{(0)}-2-(0)-1<0$

Put $x=1$ in equation (1) we get

$f(1)=e^{(1)}-2-(1)$

$f(1)=-0.2817<0$

Put $x=2$ in equation (1) we get

$f(2)=e^{(2)}-2-(2)$

$f(2)=3.3891>0$

Therefore $f(x)$ has opposite in interval $[1,2]$

b). $f(x)=\cos x+1-x$ $\quad x$ is in radians

Sol: Given $f(x)=\cos x+1-x$................(1)

Put $x=0$ in equation (1) we get

$f(0)=\cos(0)+1-0$

$f(0)=2>0$

Put $x=1$ in equation (1) we get

$f(1)=\cos(1)+1-1$

$f(1)=0.9998>0$

Put $x=2$ in equation (1) we get

$f(2)=\cos(2)+1-2$

$f(2)=-6.0917<0$

Therefore $f(x)$ has opposite in interval $[1,2]$

c). $f(x)=x^2-10x+23$

Sol: Given $f(x)=x^2-10x+23$.............(1)

Put $x=0$ in equation (1) we get

$f(0)=(0)^2-10(0)+23$

$f(0)=23>0$

Put $x=1$ in equation (1) we get

$f(1)=(1)^2-10(1)+23$

$f(1)=1-10+23$

$f(1)=14>0$

Put $x=2$ in equation (1) we get

$f(2)=(2)^2-10(2)+23$

$f(2)=4-20+23$

$f(2)=7>0$

Put $x=3$ in equation (1) we get

$f(3)=(3)^2-10(3)+23$

$f(3)=9-30+23$

$f(3)=2>0$

Put $x=4$ in equation (1) we get

$f(4)=(4)^2-10(4)+23$

$f(4)=16-40+23$

$f(4)=-1<0$

Therefore $f(x)$ has opposite in interval $[3,4]$

Q2. Compute four iterates of bisection method for following functions with indicated interval $[a_o,b_o]$

a). $f(x)=e^x-2-x \quad [1,1.8]$

Sol: Given $f(x)=e^x-2-x$.................(1)

First Iteration: Here $a_o=1,\ b_o=1.8$

Put $x=1$ in equation (1) we get

$f(1)=e^{(1)}-2-(1)$

$f(1)=-0.2818<0$

Put $x=1.8$ in equation (1) we get

$f(1.8)=e^{(1.8)}-2-(1.8)$

$f(1.8)=2.2496>0$

Therefore $f(1)<0,\ f(1.8)>0$

So the actual roots is lie in the interval $[1,1.8]$

Then $c_o=\frac{a_o+b_o}{2}$ putting the values

$c_0=\frac{1+1.8}{2}=\frac{2.8}{2}$

$c_o=1.4$

Second Iteration: Put $x=1.4$ in eq (1) we get

$f(1.4)=e^{(1.4)}-2-(1.4)$

$f(1.4)=0.6552>0$

Therefore $f(1)<0,\ f(1.4)>0$, $f(1.8)>0$

So the actual roots is lie in the interval $[1,1.4]$

Update $a_1=1,\ b_1=1.4$

Then $c_1=\frac{a_1+b_1}{2}$ putting the values

$c_1=\frac{1+1.4}{2}=\frac{2.4}{2}$

$c_1=1.2$

Third Iteration: Put $x=1.2$ in eq (1) we get

$f(1.2)=e^{(1.2)}-2-(1.2)$

$f(1.2)=0.1201>0$

Therefore $f(1)<0,\ f(1.2)>0,\ f(1.4)>0$

So the actual roots is lie in the interval $[1,1.2]$

Update $a_2=1,\ b_2=1.2$

Then $c_2=\frac{a_2+b_2}{2}$ putting the values

$c_2=\frac{1+1.2}{2}=\frac{2.2}{2}$

$c_2=1.1$

Fourth Iteration: Put $x=1.1$ in eq (1) we get

$f(1.1)=e^{(1.1)}-2-(1.1)$

$f(1.1)=-0.0958<0$

Therefore $f(1)<0,\ f(1.1)<0,\ f(1.2)>0$

So the actual roots is lie in the interval $[1.1,1.2]$

Update $a_3=1.1,\qquad b_3=1.2$

Then $c_3=\frac{a_3+b_3}{2}$ putting the values

$c_3=\frac{1.1+1.2}{2}=\frac{2.3}{2}$

$c_3=1.15$

Put $x=1.15$ in equation (1) we get

$f(1.15)=e^{(1.15)}-2-(1.15)$

$f(1.15)=0.0082>0$

Therefore $f(1.1)<0,\ f(1.15)>0$, $f(1.2)>0$

So the actual roots is lie in the interval $[1.1,1.15]$

Update $a_4=1.1,\qquad b_4=1.15$

b). $f(x)=\cos x+1-x\quad [0.8,1.6]$ x is in radians

Sol: Given $f(x)=\cos x+1-x$..............(1)

First Iteration: Here $a_o=0.8,\qquad b_o=1.6$

Put $x=0.8$ in equation (1) we get

$f(0.8)=\cos(0.8)+1-(0.8)$

$f(0.8)=0.8967>0$

Put $x=1.6$ in equation (1) we get

$f(1.6)=\cos(1.6)+1-(1.6)$

$f(1.6)=-0.6292<0$

Therefore $f(0.8)>0,\ f(1.6)<0$

So the actual roots is lie in the interval $[0.8,1.6]$

Then $c_o=\frac{a_o+b_o}{2}$ putting the values

$c_o=\frac{0.8+1.6}{2}=\frac{2.4}{2}$

$c_o=1.2$

Second Iteration: Put $x=1.2$ in eq (1) we get

$f(1.2)=\cos(1.2)+1-(1.2)$

$f(1.2)=0.1624>0$

Therefore $f(0.8)>0,\ f(1.2)>0,\ f(1.6)<0$

So the actual roots is lie in the interval $[1.2,1.6]$

Update $a_1=1.2,\qquad b_1=1.6$

Then $c_1=\frac{a_1+b_1}{2}$ putting the values

$c_1=\frac{1.2+1.6}{2}=\frac{2.8}{2}$

$c_1=1.4$

Third Iteration: Put $x=1.4$ in equation (1) we get

$f(1.4)=\cos(1.4)+1-(1.4)$

$f(1.4)=-0.2300<0$

Therefore $f(1.2)>0, f(1.4)<0, f(1.6)<0$

So the actual roots is lie in the interval $[1.2,1.4]$

Update $a_2=1.2,\qquad b_2=1.4$

Then $c_2=\frac{a_2+b_2}{2}$ putting the values

$c_2=\frac{1.2+1.4}{2}=\frac{2.6}{2}$

$c_2=1.3$

Fourth Iteration: Put $x=1.3$ in equation (1)

$f(1.3)=\cos(1.3)+1-(1.3)=-0.0325<0$

Therefore $f(1.2)>0,\ f(1.3)<0\ ,\ f(1.4)<0$

So the actual roots is lie in the interval $[1.2,1.3]$

Update $a_3=1.2,\qquad b_3=1.3$

Then $c_3=\dfrac{a_3+b_3}{2}$ putting the values

$$c_3=\frac{1.2+1.3}{2}=\frac{2.5}{2}$$
$$c_3=1.25$$

Put $x=1.25$ in equation (1) we get

$f(1.25)=\cos(1.25)+1-(1.25)$

$f(1.25)=0.0653>0$

Therefore $f(1.2)>0,\ f(1.25)>0,\ f(1.3)<0$

So the actual roots is lie in the interval $[1.25,1.3]$

Update $a_4=1.25,\qquad b_4=1.3$

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c). $f(x)=x^2-10x+23\qquad [3.2,4]$

Sol: Given $f(x)=x^2-10x+23$............(1)

First Iteration: Here $a_0=3.2,\qquad b_0=4$

Put $x=3.2$ in equation (1) we get

$f(3.2)=(3.2)^2-10(3.2)+23$

$f(3.2)=1.24>0$

Put $x=4$ in equation (1) we get

$f(4)=(4)^2-10(4)+23$

$f(4)=-1<0$

Therefore $f(3.2)>0,\ f(4)<0$

So the actual roots is lie in the interval $[3.2,4]$

Then $c_o=\dfrac{a_o+b_o}{2}$ putting the values

$$c_o=\frac{3.2+4}{2}=\frac{7.2}{2}$$
$$c_o=3.6$$

Second Iteration: Put $x=3.6$ in equation (1)

$f(3.6)=(3.6)^2-10(3.6)+23$

$f(3.6)=-0.04<0$

Therefore $f(3.2)>0,\ f(3.6)<0,\ f(4)<0$

So the actual roots is lie in the interval $[3.2,3.6]$

Update $a_1=3.2,\qquad b_1=3.6$

Then $c_1=\dfrac{a_1+b_1}{2}$ putting the values

$$c_1=\frac{3.2+3.6}{2}=\frac{6.8}{2}$$
$$c_1=3.4$$

Third Iteration: Put $x=3.4$ in equation (1) we get

$f(3.4)=(3.4)^2-10(3.4)+23=0.56>0$

Therefore $f(3.2)>0,\ f(3.4)>0,\ f(3.6)<0$

So the actual roots is lie in the interval $[3.4,3.6]$

Update $a_2=3.4,\qquad b_2=3.6$

Then $c_2=\dfrac{a_2+b_2}{2}$ putting the values

$$c_2=\frac{3.4+3.6}{2}=\frac{7}{2}$$
$$c_2=3.5$$

Fourth Iteration: Put $x=1.1$ in equation (1)

$f(3.5)=(3.5)^2-10(3.5)+23$

$f(3.5)=0.25>0$

Therefore $f(3.4)>0,\ f(3.5)>0,\ f(3.6)<0$

Update $a_3=3.5,\qquad b_3=3.6$

So the actual roots is lie in the interval $[3.5,3.6]$

Then $c_3=\dfrac{a_3+b_3}{2}$ putting the values

$$c_3=\frac{3.5+3.6}{2}=\frac{7.1}{2}$$
$$c_3=3.55$$

Put $x=3.55$ in equation (1) we get

$f(3.55)=(3.55)^2-10(3.55)+23$

$f(3.55)=0.1025>0$

Therefore $f(3.5)>0,\ f(3.55)>0,\ f(3.6)<0$

So the actual roots is lie in the interval $[3.55,3.6]$

Update $a_4=3.55,\qquad b_4=3.6$

Q3. Compute four iterates of the regula-falsi method for following functions with indicated interval $[a_o,b_o]$:

a). $f(x)=e^x-2-x\qquad [-2.4,-1.6]$

Sol: Given $f(x)=e^x-2-x$..................(1)

First iteration Here $a_0=-2.4,\qquad b_0=-1.6$

Put $x=-2.4$ in equation (1) we get

$f(-2.4)=e^{(-2.4)}-2-(-2.4)$

$f(-2.4)=0.4907>0$

Put $x=-1.6$ in equation (1) we get

$f(-1.6)=e^{(-1.6)}-2-(-1.6)$

$f(-1.6)=-0.1981<0$

Therefore $f(-2.4)>0,\ f(-1.6)<0$

So the actual roots is lie in interval $[-2.4,-1.6]$

Then $c_o=a_o-\dfrac{(a_o-b_o)f(a_o)}{f(a_o)-f(b_o)}$ putting the values

$$c_o=-2.4-\frac{(-2.4-(-1.6))(0.4907)}{0.4907-(-0.1981)}$$
$$c_o=-2.4-\frac{(-2.4+1.6)(0.4907)}{0.4907+0.1981}$$

$$c_o = -2.4 - \frac{(-0.8)(0.4907)}{0.6888}$$
$$c_o = -2.4 + 0.5699$$
$$c_o = -1.8300$$

Second Iteration: Put $x = -1.8300$ in eq (1) we get

$$f(-1.8300) = e^{(-1.8300)} - 2 - (-1.8300)$$
$$f(-1.8300) = -0.0096 < 0$$
$$\therefore \quad f(-2.4) > 0,\ f(-1.8300) < 0,\ f(-1.6) < 0$$

So actual roots is lie in interval $[-2.4, -1.8300]$

Update $a_1 = -2.4, \qquad b_1 = -1.8300$

Then $c_1 = a_1 - \frac{(a_1 - b_1) f(a_1)}{f(a_1) - f(b_1)}$ putting the values

$$c_1 = -2.4 - \frac{(-2.4 - (-1.8300))(0.4907)}{0.4907 - (-0.0096)}$$
$$c_1 = -2.4 - \frac{(-2.4 + 1.8300)(0.4907)}{0.4907 + 0.0096}$$
$$c_1 = -2.4 - \frac{(-0.57)(0.4907)}{0.5003}$$
$$c_1 = -2.4 + 0.5591$$
$$c_1 = -1.8409$$

Third Iteration: Put $x = -1.8409$ in equation (1)

$$f(-1.8409) = e^{(-1.8409)} - 2 - (-1.8409)$$
$$f(-1.8409) = -0.00043 < 0$$
$$\therefore \quad f(-2.4) > 0,\ f(-1.8409) < 0\ \ f(-1.8300) < 0$$

So actual roots is lie in interval $[-2.4, -1.8409]$

Update $a_2 = -2.4, \qquad b_2 = -1.8409$

Then $c_2 = a_2 - \frac{(a_2 - b_2) f(a_2)}{f(a_2) - f(b_2)}$ putting the values

$$c_2 = -2.4 - \frac{(-2.4 - (-1.8409))(0.4907)}{0.4907 - (-0.00043)}$$
$$c_2 = -2.4 - \frac{(-2.4 + 1.8409)(0.4907)}{0.4907 + 0.00043}$$
$$c_2 = -2.4 - \frac{(-0.5591)(0.4907)}{0.49113}$$
$$c_2 = -2.4 + 0.5586$$
$$c_2 = -1.8414$$

Fourth Iteration: Put $x = -1.8414$ in eq (1) we get

$$f(-1.8414) = e^{(-1.8414)} - 2 - (-1.8414)$$
$$f(-1.8414) = -0.0000048 < 0$$
$$\therefore \quad f(-2.4) > 0,\ f(-1.8414) < 0\ \ f(-1.8409) < 0$$

So actual roots is lie in interval $[-2.4, -1.8414]$

Update $a_3 = -2.4, \qquad b_3 = -1.8414$

Then $c_3 = a_3 - \frac{(a_3 - b_3) f(a_3)}{f(a_3) - f(b_3)}$ putting the values

$$c_3 = -2.4 - \frac{(-2.4 - (-1.8414))(0.4907)}{0.4907 - (-0.0000048)}$$
$$c_3 = -2.4 - \frac{(-2.4 + 1.8414)(0.4907)}{0.4907 + 0.0000048}$$
$$c_3 = -2.4 - \frac{(-0.5586)(0.4907)}{0.4907048}$$
$$c_3 = -2.4 + 0.5586$$
$$c_3 = -1.8414$$

Put $x = -1.8414$ in equation (1) we get

$$f(-1.8414) = e^{(-1.8414)} - 2 - (-1.8414)$$
$$f(-1.8414) = -0.0000048 < 0$$
$$\therefore \quad f(-2.4) > 0,\ f(-1.8414) < 0,\ f(-1.8414) < 0$$

So actual roots is lie in interval $[-2.4, -1.8414]$

Update $a_4 = -2.4, \qquad b_4 = -1.8414$

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b). $f(x) = \cos x + 1 - x \quad [0.8, 1.6]$ x is in radians

Sol: Given $f(x) = \cos x + 1 - x$..............(1)

First iteration: Here $a_0 = 0.8, \qquad b_0 = 1.6$

Put $x = 0.8$ in equation (1) we get

$$f(0.8) = \cos(0.8) + 1 - (0.8)$$
$$f(0.8) = 0.8967 > 0$$

Put $x = -1.6$ in equation (1) we get

$$f(1.6) = \cos(1.6) + 1 - (1.6)$$
$$f(1.6) = -0.6292 < 0$$

Therefore $f(0.8) > 0,\ f(1.6) < 0$

So the actual roots is lie in the interval $[0.8, 1.6]$

Then $c_o = a_o - \frac{(a_o - b_o) f(a_o)}{f(a_o) - f(b_o)}$ putting the values

$$c_o = 0.8 - \frac{(0.8 - 1.6)(0.8967)}{0.8967 - (-0.6292)}$$
$$c_o = 0.8 - \frac{(-0.8)(0.8967)}{0.8967 + 0.6292}$$
$$c_o = 0.8 + \frac{0.71736}{1.5259}$$
$$c_o = 0.8 + 0.4701$$
$$c_o = 1.2701$$

Second Iteration: Put $x = 1.2701$ in equation (1)

$$f(1.2701) = \cos(1.2701) + 1 - (1.2701)$$
$$f(1.2701) = 0.0261 > 0$$

Therefore $f(0.8) > 0,\ f(1.2701) > 0,$
$f(1.6) < 0$

So the actual roots is lie in interval $[1.2701, 1.6]$

Update $a_1 = 1.2701, \quad b_1 = 1.6$

Then $c_1 = a_1 - \frac{(a_1 - b_1) f(a_1)}{f(a_1) - f(b_1)}$ putting the values

$c_1 = 1.2701 - \frac{(1.2701-1.6)(0.0261)}{0.0261-(-0.6292)}$

$c_1 = 1.2701 - \frac{(-0.3299)(0.0261)}{0.0261+0.6292}$

$c_1 = 1.2701 + \frac{0.00861039}{0.6553}$

$c_1 = 1.2701 + 0.0131$

$c_1 = 1.2832$

Third Iteration: Put $x = 1.2832$ in equation (1)

$f(1.2832) = \cos(1.2832) + 1 - (1.2832)$

$f(1.2832) = 0.000448 > 0$

$\therefore \ f(1.2701) > 0, \ f(1.2832) > 0, \ f(1.6) < 0$

So the actual roots is lie in interval $[1.2832, 1.6]$

Update $a_2 = 1.2832, \quad b_2 = 1.6$

Then $c_2 = a_2 - \frac{(a_2 - b_2) f(a_2)}{f(a_2) - f(b_2)}$ putting the values

$c_2 = 1.2832 - \frac{(1.2832-1.6)(0.000448)}{0.000448-(-0.6292)}$

$c_2 = 1.2832 - \frac{(-3.3168)(0.000448)}{0.000448+0.6292}$

$c_2 = 1.2832 + \frac{0.000142}{0.6296}$

$c_2 = 1.2832 + 0.0002255$

$c_2 = 1.2834$

Fourth Iteration: Put $x = 1.2834$ in equation (1)

$f(1.2834) = \cos(1.2834) + 1 - (1.2834)$

$f(1.2834) = 0.0000563 > 0$

$\therefore \ f(1.2832) > 0, \ f(1.2834) > 0, \ f(1.6) < 0$

So the actual roots is lie in interval $[1.2834, 1.6]$

Update $a_3 = 1.2834, \quad b_3 = 1.6$

Then $c_3 = a_3 - \frac{(a_3 - b_3) f(a_3)}{f(a_3) - f(b_3)}$ putting the values

$c_3 = 1.2834 - \frac{(1.2834-1.6)(0.0000563)}{0.0000563-(-0.6292)}$

$c_3 = 1.2834 - \frac{(-3.3166)(0.0000563)}{0.0000563+0.6292}$

$c_3 = 1.2834 + \frac{0.0000178}{0.6292563}$

$c_3 = 1.2834 + 0.0000283$

$c_3 = 1.2834283$

Put $x = 1.2834$ in equation (1) we get

$f(1.2834) = \cos(1.2834) + 1 - (1.2834)$

$f(1.2834) = 0.0000563 > 0$

Therefore $f(1.2834) > 0,$

$f(1.2834) > 0, f(1.6) < 0$

So the actual roots is lie in interval $[1.2834, 1.6]$

Update $a_4 = 1.2834, \quad b_4 = 1.6$

c). $f(x) = x^2 - 10x + 23 \qquad [3.2, 4]$

Sol: Given $f(x) = x^2 - 10x + 23$...............(1)

First iteration: Here $a_0 = 3.2, \qquad b_0 = 4$

Put $x = 3.2$ in equation (1) we get

$f(3.2) = (3.2)^2 - 10(3.2) + 23$

$f(3.2) = 1.24 > 0$

Put $x = 4$ in equation (1) we get

$f(4) = (4)^2 - 10(4) + 23$

$f(4) = -1 < 0$

Therefore $f(3.2) > 0, \ f(4) < 0$

So the actual roots is lie in the interval $[3.2, 4]$

Then $c_o = a_o - \frac{(a_o - b_o) f(a_o)}{f(a_o) - f(b_o)}$ putting the values

$c_o = 3.2 - \frac{(3.2-4)(1.24)}{1.24-(-1)}$

$c_o = 3.2 - \frac{(-0.8)(1.24)}{1.24+1}$

$c_o = 3.2 + \frac{0.992}{2.24}$

$c_o = 3.2 + 0.4429$

$c_o = 3.6429$

Second Iteration: Put $x = 3.6429$ in eq (1) we get

$f(3.6429) = (3.6429)^2 - 10(3.6429) + 23$

$f(3.6429) = -0.1583 < 0$

Therefore $f(3.2) > 0, \ f(3.6429) < 0, f(4) < 0$

So actual roots is lie in the interval $[3.2, 3.6429]$

Update $a_1 = 3.2, \qquad b_1 = 3.6429$

Then $c_1 = a_1 - \frac{(a_1 - b_1) f(a_1)}{f(a_1) - f(b_1)}$ putting the values

$c_1 = 3.2 - \frac{(3.2-3.6429)(1.24)}{1.24-(-0.1583)}$

$c_1 = 3.2 - \frac{(-0.4429)(1.24)}{1.24+0.1583}$

$c_1 = 3.2 + \frac{0.549196}{1.3983}$

$c_1 = 3.2 + 0.3928$

$c_1 = 3.5928$

Third Iteration: Put $x = 3.5928$ in equation (1)

$f(3.5928) = (3.5928)^2 - 10(3.5928) + 23$

$f(3.5928) = -0.0198 < 0$

$\therefore \ f(3.2) > 0, \ f(3.5928) < 0, \ f(3.6429) < 0$

So the actual roots is lie in interval $[3.2, 3.5928]$

Update $a_2 = 3.2, \qquad b_2 = 3.5928$

Then $c_2 = a_2 - \frac{(a_2 - b_2) f(a_2)}{f(a_2) - f(b_2)}$ putting the values

$$c_2 = 3.2 - \frac{(3.2 - 3.5928)(1.24)}{1.24 - (-0.0198)}$$
$$c_2 = 3.2 - \frac{(3.3928)(1.24)}{1.24 + 0.0198}$$
$$c_2 = 3.2 + \frac{0.4871}{1.2598}$$
$$c_2 = 3.2 + 0.3866$$
$$c_2 = 3.5866$$

Fourth Iteration: Put $x = 3.5866$ in equation (1)

$$f(3.5866) = (6.5866)^2 - 10(3.5866) + 23$$
$$f(3.5866) = -0.0023 < 0$$
$$\therefore \; f(3.2) > 0,\; f(3.5866) < 0,\; f(3.5928) < 0$$

So the actual roots is lie in interval $[3.2, 3.5866]$

Update $a_3 = 3.2, \quad b_3 = 3.5866$

Then $c_3 = a_3 - \frac{(a_3 - b_3) f(a_3)}{f(a_3) - f(b_3)}$ putting the values

$$c_3 = 3.2 - \frac{(3.2 - 3.5866)(1.24)}{1.24 - (-0.0023)}$$
$$c_3 = 3.2 - \frac{(-0.3866)(1.24)}{1.24 + 0.0023}$$
$$c_3 = 3.2 + \frac{0.4793}{1.2423}$$
$$c_3 = 3.2 + 0.3859$$
$$c_3 = 3.5859$$

Put $x = 3.5859$ in equation (1) we get

$$f(3.5859) = (3.5859)^2 - 10(3.5859) + 23$$
$$f(3.5859) = -0.00032119 < 0$$
$$\therefore \; f(3.2) > 0,\; f(3.5859) < 0,\; f(3.5866) < 0$$

So the actual roots is lie in interval $[3.2, 3.5866]$

Update $a_4 = 3.2, \quad b_4 = 3.5859$

Q4. Let $f(x) = x^2 - 2x - 1$ The actual roots of $f(x)$ are $r_1 = 2.414214$ and $r_2 = -0.414214$ use Newton-Raphson method

a). with initial start $x_o = 2.5$ to compute iterates x_1, x_2 and x_3 that will converge actual root r_1 of $f(x)$

Sol: Given $f(x) = x^2 - 2x - 1$..................(1)

Differentiating with respect to x, we get

$$\tfrac{d}{dx} f(x) = \tfrac{d}{dx} x^2 - 2\tfrac{d}{dx} x - \tfrac{d}{dx} 1$$
$$f'(x) = 2x \tfrac{d}{dx} x - 2(1) - 0$$
$$f'(x) = 2x - 2 \text{.......................................}(2)$$

Newton Rephson formula is given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Putting the value of $f(x)$ and $f'(x)$ we have

$$x_{i+1} = x_i - \frac{x_i^2 - 2x_i - 1}{2x_i - 2} \text{.......................................}(A)$$

For First iteration put $i = 0$ in equation (A), we get

$$x_{0+1} = x_0 - \frac{x_0^2 - 2x_0 - 1}{2x_0 - 2} \qquad \text{Put } x_o = 2.5$$
$$x_1 = 2.5 - \frac{(2.5)^2 - 2(2.5) - 1}{2(2.5) - 2}$$
$$x_1 = 2.5 - \frac{6.25 - 5 - 1}{5 - 2}$$
$$x_1 = 2.5 - \frac{0.25}{3}$$
$$x_1 = 2.5 - 0.0833333$$
$$x_1 = 2.4166667$$

For Second iteration put $i = 1$ in equation (A),

$$x_{1+1} = x_1 - \frac{x_1^2 - 2x_1 - 1}{2x_1 - 2} \qquad \text{Put } x_1 = 2.4166667$$
$$x_2 = 2.4166667 - \frac{(2.4166667)^2 - 2(2.4166667) - 1}{2(2.4166667) - 2}$$
$$x_2 = 2.4166667 - \frac{5.8402778 - 4.8333333 - 1}{4.83333333 - 2}$$
$$x_2 = 2.4166667 - \frac{0.0069444}{2.8333333}$$
$$x_2 = 2.4166667 - 0.002451$$
$$x_2 = 2.4142157$$

For Third iteration put $i = 2$ in equation (A),

$$x_{2+1} = x_2 - \frac{x_2^2 - 2x_2 - 1}{2x_2 - 2} \qquad \text{Put } x_2 = 2.4142157$$
$$x_3 = 2.4142157 - \frac{(2.4142157)^2 - 2(2.4142157) - 1}{2(2.414215) - 2}$$
$$x_3 = 2.4142157 - \frac{5.8284374 - 4.8284314 - 1}{4.8284314 - 2}$$
$$x_3 = 2.4142157 - \frac{0.000006}{2.8284314}$$
$$x_3 = 2.4142157 - 0.0000021$$
$$x_3 = 2.4142136$$

This is nearly equal to $r_1 = 2.414214$

---

b). with initial start $x_o = -0.5$ to compute iterates x_1, x_2 & x_3 that will converge actual root r_2 of $f(x)$

Sol: Given $f(x) = x^2 - 2x - 1$..................(1)

Differentiating with respect to x, we get

$$\tfrac{d}{dx} f(x) = \tfrac{d}{dx} x^2 - 2\tfrac{d}{dx} x - \tfrac{d}{dx} 1$$
$$f'(x) = 2x \tfrac{d}{dx} x - 2(1) - 0$$
$$f'(x) = 2x - 2 \text{.......................................}(2)$$

Newton Rephson formula is given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Putting the value of $f(x)$ and $f'(x)$ we have

$$x_{i+1}=x_i-\frac{x_i^2-2x_i-1}{2x_i-2}\dots\dots\dots(A)$$

For First iteration put $i=0$ in equation (A), we get

$$x_{0+1}=x_0-\frac{x_0^2-2x_0-1}{2x_0-2}\qquad \text{Put } x_o=-0.5$$

$$x_1=-0.5-\frac{(-0.5)^2-2(-0.5)-1}{2(-0.5)-2}$$

$$x_1=-0.5-\frac{0.25+1-1}{-1-2}$$

$$x_1=-0.5-\frac{0.25}{-3}$$

$$x_1=-0.5+0.0833333$$

$$x_1=-0.4166667$$

For Second iteration put $i=1$ in equation (A),

$$x_{1+1}=x_1-\frac{x_1^2-2x_1-1}{2x_1-2}\quad \text{Put } x_1=-0.4166667$$

$$x_2=-0.4166667-\frac{(-0.4166667)^2-2(-0.4166667)-1}{2(-0.4166667)-2}$$

$$x_2=-0.4166667-\frac{0.1736111+0.8333333-1}{-0.83333333-2}$$

$$x_2=-0.4166667-\frac{0.0069444}{-2.8333333}$$

$$x_2=-0.4166667+0.002451$$

$$x_2=-0.4142157$$

For Third iteration put $i=2$ in equation (A),

$$x_{2+1}=x_2-\frac{x_2^2-2x_2-1}{2x_2-2}\quad \text{Put } x_2=-0.4142157$$

$$x_3=-0.4142157-\frac{(-0.4142157)^2-2(-0.4142157)-1}{2(-0.414215)-2}$$

$$x_3=-0.4142157-\frac{0.1715729+0.8284314-1}{-0.8284314-2}$$

$$x_3=-0.4142157-\frac{0.000006}{-2.8284314}$$

$$x_3=-0.4142157+0.0000021$$

$$x_3=-0.4142136$$

This is nearly equal to $r_2=-0.414214$

Q5. Find iterate x_3 of the Newton-Raphson iterative method for the following functions with initial start x_o

a). $f(x)=x^3-3,\qquad x_o=1$

Sol: Given $f(x)=x^3-3\dots\dots\dots(1)$

Differentiating with respect to x, we get

$$\tfrac{d}{dx}f(x)=\tfrac{d}{dx}x^3-\tfrac{d}{dx}3$$

$$f'(x)=3x^2\tfrac{d}{dx}x-0$$

$$f'(x)=3x^2\dots\dots\dots(2)$$

Newton Rephson formula is given by

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

Putting the value of $f(x)$ and $f'(x)$ we have

$$x_{i+1}=x_i-\frac{x_i^3-3}{3x_i^2}\dots\dots\dots(A)$$

For First iteration put $i=0$ in equation (A), we get

$$x_{0+1}=x_0-\frac{x_0^3-3}{3x_o^2}\qquad \text{Put } x_o=1$$

$$x_1=1-\frac{(1)^3-3}{3(1)^2}$$

$$x_1=1-\frac{1-3}{3}$$

$$x_1=1-\frac{-2}{3}=1+\frac{2}{3}$$

$$x_1=1.6666667$$

For Second iteration put $i=1$ in equation (A),

$$x_{1+1}=x_1-\frac{x_1^3-3}{3x_1^2}\quad \text{Put } x_1=1.6666667$$

$$x_2=1.6666667-\frac{(1.6666667)^3-3}{3(1.6666667)^2}$$

$$x_2=1.6666667-\frac{4.6296296-3}{8.3333333}$$

$$x_2=1.6666667-\frac{1.6296296}{8.3333333}$$

$$x_2=1.6666667-0.1955555$$

$$x_2=1.4711111$$

For Third iteration put $i=2$ in equation (A),

$$x_{2+1}=x_2-\frac{x_2^3-3}{3x_2^2}\quad \text{Put } x_2=1.417111111$$

$$x_3=1.4711111-\frac{(1.4711111)^3-3}{3(1.4711111)^2}$$

$$x_3=1.4711111-\frac{3.1837314-3}{6.4925037}$$

$$x_3=1.4711111-\frac{0.1837314}{6.4925037}$$

$$x_3=1.4711111-0.028299$$

$$x_3=1.4428121$$

b). $f(x)=\sin(x),\qquad x_o=1$

Sol: Given $f(x)=\sin(x)\dots\dots\dots(1)$

Differentiating with respect to x, we get

$$\tfrac{d}{dx}f(x)=\tfrac{d}{dx}\sin(x)$$

$$f'(x)=\cos(x)\tfrac{d}{dx}x$$

$$f'(x)=\cos(x)\dots\dots\dots(2)$$

Newton Rephson formula is given by

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

Putting the value of $f(x)$ and $f'(x)$ we have

$$x_{i+1}=x_i-\frac{\sin(x_i)}{\cos(x_i)}$$

$$x_{i+1}=x_i-\tan(x_i)\dots\dots\dots(A)$$

For First iteration put $i=0$ in equation (A), we get

$x_{0+1}=x_0-\tan(x_o)$ Put $x_o=1$

$x_1=1-\tan(1)$

$x_1=1-1.5574077$

$x_1=-0.5574077$

For Second iteration put $i=1$ in equation (A)

$x_{1+1}=x_1-\tan(x_1)$ Put $x_1=-0.5574077$

$x_2=-0.5574077-\tan(-0.5574077)$

$x_2=-0.5574077-(-0.6233442)$

$x_2=0.0659365$

For Third iteration put $i=2$ in equation (A)

$x_{2+1}=x_2-\tan(x_2)$ Put $x_2=0.0659365$

$x_3=0.0659365-\tan(0.0659365)$

$x_3=0.0659365-0.0660322$

$x_3=-0.0000957$

c). $f(x)=x^3+2x-1,\qquad x_o=0$

Sol: Given $f(x)=x^3+2x-1$................(1)

Differentiating with respect to x, we get

$\frac{d}{dx}f(x)=\frac{d}{dx}x^3+2\frac{d}{dx}x-\frac{d}{dx}1$

$f'(x)=3x^2\frac{d}{dx}x+2(1)-0$

$f'(x)=3x^2+2$..(2)

Newton Rephson formula is given by

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

Putting the value of $f(x)$ and $f'(x)$ we have

$$x_{i+1}=x_i-\frac{x_i^3+2x_i-1}{3x_i^2+2}\text{.......................................}(A)$$

For First iteration put $i=0$ in equation (A), we get

$x_{0+1}=x_0-\frac{x_0^3+2x_0-1}{3x_0^2+2}$ Put $x_o=0$

$x_1=0-\frac{(0)^3+2(0)-1}{3(0)^2+2}$

$x_1=0-\frac{-1}{2}$

$x_1=\frac{1}{2}$

$x_1=0.5$

For Second iteration put $i=1$ in equation (A)

$x_{1+1}=x_1-\frac{x_1^3+2x_1-1}{3x_1^2+2}$ Put $x_1=0.5$

$x_2=0.5-\frac{(0.5)^3+2(0.5)-1}{3(0.5)^2+2}$

$x_2=0.5-\frac{0.125+1-1}{0.75+2}$

$x_2=0.5-\frac{0.125}{2.75}$

$x_2=0.5-0.0454545$

$x_2=0.45454545$

For Third iteration put $i=2$ in equation (A)

$x_{2+1}=x_2-\frac{x_2^3+2x_2-1}{3x_2^2+2}$ Put $x_2=0.45454545$

$x_3=0.45454545-\frac{(0.45454545)^3+2(0.45454545)-1}{3(0.45454545)^2+2}$

$x_3=0.45454545-\frac{0.0939144+0.909090-1}{0.6198347+2}$

$x_3=0.454545-\frac{0.0030053}{2.6198347}$

$x_3=0.454545-0.0011471$

$x_3=0.4533983$

Q6. Use Newton-Raphson iterative method to approximate the actual r of the following non-linear equations with indicated interval: Continue the process until two consecutive iterates will agree to three decimal places.

a) $f(x)=x^3+3x-1=0\qquad on\qquad (0,1)$

Sol: Given $f(x)=x^3+3x-1=0$...........(1)

Differentiating with respect to x, we get

$\frac{d}{dx}f(x)=\frac{d}{dx}x^3+3\frac{d}{dx}x-\frac{d}{dx}1$

$f'(x)=3x^2\frac{d}{dx}x+3(1)-0$

$f'(x)=3x^2+3$..(2)

Newton Rephson formula is given by

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

Putting the value of $f(x)$ and $f'(x)$ we have

$$x_{i+1}=x_i-\frac{x_i^3+3x_i-1}{3x_i^2+3}\text{.......................................}(A)$$

For First iteration put $i=0$ in equation (A), we get

$x_{0+1}=x_0-\frac{x_0^3+3x_0-1}{3x_0^2+3}$ Put $x_o=0$

$x_1=0-\frac{(0)^3+3(0)-1}{3(0)^2+3}$

$x_1=0-\frac{-1}{3}=\frac{1}{3}$

$x_1=0.3333333$

For Second iteration put $i=1$ in equation (A)

$x_{1+1}=x_1-\frac{x_1^3+3x_1-1}{3x_1^2+3}$ Put $x_1=0.3333333$

$x_2=0.3333333-\frac{(0.3333333)^3+3(0.3333333)-1}{3(0.3333333)^2+3}$

$x_2=0.3333333-\frac{0.0370370}{3.3333333}$

$x_2=0.3333333-0.011111111$

$x_2=0.3222222$

For Third iteration put $i=2$ in equation (A)

$$x_{2+1}=x_2-\frac{x_2^3+3x_2-1}{3x_2^2+3}\quad \text{Put } x_2=0.3222222$$

$$x_3=0.3222222-\frac{(0.3222222)^3+3(0.3222222)-1}{3(0.3222222)^2+3}$$

$$x_3=0.3222222-\frac{0.0001221}{3.3114815}$$

$$x_3=0.3222222-0.0000369$$

$$x_3=0.3221854$$

For <u>Fourth iteration</u> put $i=3$ in equation (A)

$$x_{3+1}=x_3-\frac{x_3^3+3x_3-1}{3x_3^2+3}\quad \text{Put } x_3=0.3221854$$

$$x_4=0.3221854-\frac{(0.3221854)^3+3(0.3221854)-1}{3(0.3221854)^2+3}$$

$$x_4=0.3221854-\frac{0}{3.3144102}$$

$$x_4=0.3221854-0$$

$$x_4=0.3221854$$

b) $f(x)=x^3+2x^2-x+1=0 \quad on \quad (-3,-2)$

Sol: Given $f(x)=x^3+2x^2-x+1=0......(1)$

Differentiating with respect to x, we get

$$f(x)=x^3+2x^2-x+1$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}x^3+2\frac{d}{dx}x^2-\frac{d}{dx}x+\frac{d}{dx}1$$

$$f'(x)=3x^2\frac{d}{dx}x+2(2x)\frac{d}{dx}x-1+0$$

$$f'(x)=3x^2+4x-1\text{...}(2)$$

Newton Rephson's formula is given by

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

Putting the value of $f(x)$ and $f'(x)$ we have

$$x_{i+1}=x_i-\frac{x_i^3+2x_i^2-x_i+1}{3x_i^2+4x_i-1}\text{...}(A)$$

For <u>First iteration</u> put $i=0$ in equation (A), we get

$$x_{0+1}=x_0-\frac{x_0^3+2x_0^2-x_o+1}{3x_0^2+4x_0-1}\quad \text{Put } x_o=-3$$

$$x_1=-3-\frac{(-3)^3+2(-3)^2-(-3)+1}{3(-3)^2+4(-3)-1}$$

$$x_1=-3-\frac{-5}{14}=-2.6428571$$

For <u>Second iteration</u> put $i=1$ in equation (A),

$$x_{1+1}=x_1-\frac{x_1^3+2x_1^2-x_1+1}{3x_1^2+4x_1-1}\quad \text{Put } x_1=-2.6428571$$

$$x_2=-2.6428571-\frac{(-2.6428571)^3+2(-2.6428571)^2-(-2.6428571)+1}{3(-2.6428571)^2+4(-2.6428571)-1}$$

$$x_2=-2.6428571-\frac{-0.8473032}{9.3826531}$$

$$x_2=-2.6428571+0.0903053$$

$$x_2=-2.5525519$$

For <u>Third iteration</u> put $i=2$ in equation (A)

$$x_{2+1}=x_2-\frac{x_2^3+2x_2^2-x_2+1}{3x_2^2+4x_2-1}\quad \text{Put } x_2=-2.5525519$$

$$x_3=-2.5525519-\frac{(-2.5525519)^3+2(-2.5525519)^2-(-2.5525519)+1}{3(-2.5525519)^2+4(-2.5525519)-1}$$

$$x_3=-2.5525519-\frac{-0.0476113}{8.3363555}$$

$$x_3=-2.5525519+0.0057113$$

$$x_3=-2.5468406$$

For <u>Fourth iteration</u> put $i=3$ in equation (A)

$$x_{3+1}=x_3-\frac{x_3^3+2x_3^2-x_3+1}{3x_3^2+4x_3-1}\quad \text{Put } x_3=-2.5468406$$

$$x_4=-2.5468406-\frac{(-2.5468406)^3+2(-2.5468406)^2-(-2.5468406)+1}{3(-2.5468406)^2+4(-2.5468406)-1}$$

$$x_4=-2.5468406-\frac{-0.0001847}{8.2718287}$$

$$x_4=-2.5468406+0.0000223$$

$$x_4=-2.5468183$$

c). $\sqrt[3]{x-3}=x+1 \quad on \quad [-3,-2]$

Sol: Given $\sqrt[3]{x-3}=x+1$

Or $\sqrt[3]{x-3}-x-1=0$

Let $f(x)=\sqrt[3]{x-3}-x-1=0$

Differentiating with respect to x, we get

$$\frac{d}{dx}f(x)=\frac{d}{dx}(x-3)^{\frac{1}{3}}-\frac{d}{dx}x-\frac{d}{dx}1$$

$$f'(x)=\tfrac{1}{3}(x-3)^{\frac{-2}{3}}-1-0$$

$$f'(x)=\tfrac{1}{3}(x-3)^{\frac{-2}{3}}-1\text{...}(2)$$

Newton Rephson formula is given by

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

Putting the value of $f(x)$ and $f'(x)$ we have

$$x_{i+1}=x_i-\frac{(x_i-3)^{\frac{1}{3}}-x_i-1}{\frac{1}{3}(x_i-3)^{\frac{-2}{3}}-1}\text{.....................................}(A)$$

For <u>First iteration</u> put $i=0$ in equation (A), we get

$$x_{0+1}=x_0-\frac{(x_0-3)^{\frac{1}{3}}-x_0-1}{\frac{1}{3}(x_0-3)^{\frac{-2}{3}}-1}\quad \text{Put } x_o=-3$$

$$x_{0+1}=-3-\frac{(-3-3)^{\frac{1}{3}}-(-3)-1}{\frac{1}{3}(-3-3)^{\frac{-2}{3}}-1}$$

$$x_1=-3-\frac{0.1828794}{-0.8990488}$$

$$x_1=-2.79568568$$

For <u>Second iteration</u> put $i=1$ in equation (A)

$$x_{1+1}=x_1-\frac{(x_1-3)^{\frac{1}{3}}-x_1-1}{\frac{1}{3}(x_1-3)^{\frac{-2}{3}}-1}\quad \text{Put } x_1=-2.79568568$$

$$x_{1+1}=-2.795685-\frac{(-2.795685-3)^{\frac{1}{3}}-(-2.795685)-1}{\frac{1}{3}(-2.79568568-3)^{\frac{-2}{3}}-1}$$

$$x_2=-2.79568568-\frac{0.00023652869}{-0.896700733}$$

$$x_2=-2.795421903$$

For <u>Third iteration</u> put $i=2$ in equation (A)

$$x_{2+1}=x_2-\frac{(x_2-3)^{\frac{1}{3}}-x_2-1}{\frac{1}{3}(x_2-3)^{\frac{-2}{3}}-1}$$ Put $x_2=-2.795421903$

$$x_{2+1}=x_2-\frac{(x_2-3)^{\frac{1}{3}}-x_2-1}{\frac{1}{3}(x_2-3)^{\frac{-2}{3}}-1}$$

$$x_3=-2.79542-\frac{(-2.79542-3)^{\frac{1}{3}}-(-2.79542)-1}{\frac{1}{3}(-2.7954219-3)^{\frac{-2}{3}}-1}$$

$$x_3=-2.7954219-\frac{0.00080702291}{-0.896686904}$$

$$x_3=-2.794521895$$

Q7. Can Newton-Rephson iterative method by used for the following functions?

a). $f(x)=9x^4-16x^3-36x^2+96x-60,\quad x_o=\frac{4}{3}$

Sol: Given $f(x)=9x^4-16x^3-36x^2+96x-60$

Differentiating with respect to x, we get

$$\tfrac{d}{dx}f(x)=9\tfrac{d}{dx}x^4-16\tfrac{d}{dx}x^3-36\tfrac{d}{dx}x^2+96\tfrac{d}{dx}x-\tfrac{d}{dx}60$$

$$f'(x)=9(4x^3)\tfrac{d}{dx}x-16(3x^2)\tfrac{d}{dx}x-36(2x)\tfrac{d}{dx}x+96-0$$

$$f'(x)=36x^3-48x^2-72x+96\text{..........................}(2)$$

Put $x_0=\frac{4}{3}$

$$f'\left(\frac{4}{3}\right)=36\left(\frac{4}{3}\right)^3-48\left(\frac{4}{3}\right)^2-72\left(\frac{4}{3}\right)+96$$

$$f'\left(\frac{4}{3}\right)=36\left(\frac{64}{27}\right)-48\left(\frac{16}{9}\right)-24(4)+96$$

$$f'\left(\frac{4}{3}\right)=36\left(\frac{64}{27}\right)-48\left(\frac{16}{9}\right)-24(4)+96$$

$$f'\left(\frac{4}{3}\right)=\frac{256}{3}-\frac{256}{3}-96+96$$

$$f'\left(\frac{4}{3}\right)=0$$

Thus Newton Raphson method fails, because when we put in formula

$$x_{i+1}=x_i-\frac{f(x_i)}{0}$$

It will become undefined, so we stop the process

b). $f(x)=1-\frac{1}{x},\qquad x_o=2$

Sol: Given $f(x)=1-\frac{1}{x},\qquad x_o=2$

Or $$f(x)=\frac{x-1}{x}$$

Differentiating with respect to x, we get

$$\tfrac{d}{dx}f(x)=\tfrac{d}{dx}1-\tfrac{d}{dx}x^{-1}$$

$$f'(x)=0-(-x^{-2})\tfrac{d}{dx}x$$

$$f'(x)=x^{-2}\text{...}(2)$$

Newton Rephson formula is given by

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

Putting the value of $f(x)$ and $f'(x)$ we have

$$x_{i+1}=x_i-\frac{x_i-1}{x_i.x_i^{-2}}$$

$$x_{i+1}=x_i-\frac{x_i-1}{x_i^{-1}}$$

$$x_{i+1}=x_i-x_i(x_i-1)$$

$$x_{i+1}=x_i-x_i^2+x_i$$

$$x_{i+1}=2x_i-x_i^2\text{..}(A)$$

For First iteration put $i=0$ in equation (A), we get

$x_{0+1}=2x_0-x_0^2$ Put $x_o=2$

$$x_1=2(2)-(2)^2$$

$$x_1=4-4$$

$$x_1=0$$

For Second iteration put $i=1$ in equation (A)

$x_{1+1}=2x_1-x_1^2$ Put $x_1=0$

$$x_2=2(0)-(0)^2$$

$$x_2=0$$

Now we cannot proceed any more

Because every iteration has become same answer

Trapezoidal rule:

$$I=\int_a^b f(x)\,dx\approx T_n=\frac{\Delta x}{2}[f_o+2f_1+2f_2+\cdots+2f_{n-1}+f_n]$$

Where $\Delta x=\dfrac{b-a}{n}$ and $a=x_o$, $x_n=x_o+n.\Delta x$,

and $x_n=b$ which is last knot

We find $x_o,x_1,x_2,...,x_{n-1},x_n$ and their corresponding values $f_o,f_1,f_2,...,f_{n-1},f_n$

Simpson's rule:

$$I=\int_a^b f(x)\,dx\approx S_{2n}=\frac{\Delta x}{3}[f_o+4(f_1+f_3+\cdots+f_{2n-1})+2(f_2+f_4+\cdots+f_{2n-2})+f_{2n}]$$

Where $\Delta x=\dfrac{b-a}{2n}$ and $a=x_o$, $x_n=x_o+n.\Delta x$,

and $x_{2n}=b$ which is last knot

We find $x_o,x_1,x_2,...,x_{2n-1},x_{2n}$ and their corresponding values $f_o,f_1,f_2,...,f_{2n-1},f_{2n}$

Exercise 11.2

Q1. Use the trapezoidal rule to approximate the value of each definite integral. Round the answer to the nearest hundredth and compare your result with the exact value of the definite integral:

a). $I=\int_1^3 x^2\,dx\qquad n=4$

Sol: Given $I=\int_1^3 x^2\,dx\qquad n=4$

Compare with $I=\int_a^b f(x)\,dx$ we get

$a=1,\qquad b=3,\qquad f(x)=x^2$

Now $\Delta x=\frac{b-a}{n}$ putting the values

$\Delta x=\frac{3-1}{4}=\frac{2}{4} \quad \Rightarrow \quad \Delta x=\frac{1}{2}=0.5$

Put $i=0$ corresponding value of function

$x_o=a=1 \qquad f_o=f(x_o)=1^2=1$

Put $i=1$ corresponding value of function

$x_1=x_o+1.\Delta x$

$x_1=1+1(0.5) \qquad f_1=f(x_1)=1.5^2$

$x_1=1+0.5 \qquad f_1=f(1.5)=2.25$

$x_1=1.5$

Put $i=2$ corresponding value of function

$x_2=x_o+2.\Delta x$

$x_2=1+2(0.5) \qquad f_2=f(x_2)=2^2$

$x_2=1+1 \qquad f_2=f(2)=4$

$x_2=2$

Put $i=3$ corresponding value of function

$x_3=x_o+3.\Delta x$

$x_3=1+3(0.5) \qquad f_3=f(x_3)=2.5^2$

$x_3=1+1.5 \qquad f_3=f(2.5)=6.25$

$x_3=2.5$

Put $i=4$ corresponding value of function

$x_4=x_o+4.\Delta x$

$x_4=1+4(0.5) \qquad f_4=f(x_4)=3^2$

$x_4=1+2 \qquad f_4=f(3)=9$

$x_4=3=b$

Trapezoidal rule for $n=4$

$I=\int_1^3 x^2 \, dx \approx T_4=\frac{\Delta x}{2}[f_o+2f_1+2f_2+2f_3+f_4]$

putting

$I\approx\frac{0.5}{2}[1+2(2.25)+2(4)+2(6.25)+9]$

$I\approx 0.25[1+4.5+8+12.5+9]$

$I\approx 0.25[35]$

$I\approx 8.75$

$Exact=\int_1^3 x^2 \, dx$

$E=\left|\frac{x^3}{3}\right|_1^3=\frac{1}{3}[3^3-1^3]$

$E=\frac{1}{3}[27-1]=\frac{1}{3}[26]$

$E=8.67$

Error = Exact – Approximate

Error = 8.67 – 8.75

Error = 0.08

---

b). $I=\int_0^1\left(\frac{x^2}{2}+1\right)dx \qquad n=4$

Sol: Given $I=\int_0^1\left(\frac{x^2}{2}+1\right)dx \qquad n=4$

Compare with $I=\int_a^b f(x)\, dx$ we get

$a=0, \qquad b=1, \qquad f(x)=\frac{x^2}{2}+1$

Now $\Delta x=\frac{b-a}{n}$ putting the values

$\Delta x=\frac{1-0}{4} \quad \Rightarrow \quad \Delta x=\frac{1}{4}=0.25$

Put $i=0$ corresponding value of function

$f_o=f(x_o)=\frac{0^2}{2}+1$

$x_o=a=0 \qquad f_o=f(0)=0+1$

$f_o=f(0)=1$

Put $i=1$ corresponding value of function

$x_1=x_o+1.\Delta x \qquad f_1=f(x_1)=\frac{0.25^2}{2}+1$

$x_1=0+1(0.25) \qquad f_1=f(0.25)=0.03125+1$

$x_1=0.25 \qquad f_1=f(0.25)=1.03125$

Put $i=2$ corresponding value of function

$x_2=x_o+2.\Delta x \qquad f_2=f(x_2)=\frac{0.5^2}{2}+1$

$x_2=0+2(0.25) \qquad f_2=f(0.5)=0.125+1$

$x_2=0+0.5 \qquad f_2=f(0.5)=1.125$

$x_2=0.5$

Put $i=3$ corresponding value of function

$x_3=x_o+3.\Delta x \qquad f_3=f(x_3)=\frac{0.75^2}{2}+1$

$x_3=0+3(0.25) \qquad f_3=f(0.75)=0.28125+1$

$x_3=0.75 \qquad f_3=f(0.75)=1.28125$

Put $i=4$ corresponding value of function

$x_4=x_o+4.\Delta x \qquad f_4=f(x_4)=\frac{1^2}{2}+1$

$x_4=0+4(0.25) \qquad f_4=f(1)=0.5+1$

$x_4=1=b \qquad f_4=f(1)=1.5$

Trapezoidal rule for $n=4$

$I\approx T_4=\frac{\Delta x}{2}[f_o+2f_1+2f_2+2f_3+f_4]$ putting

$I\approx\frac{0.25}{2}[1+2(1.03125+1.125+1.28125)+1.5]$

$I\approx 0.125[9.375]=1.171875$

$Exact=\int_0^1\left(\frac{x^2}{2}+1\right)dx$

$E=\left|\frac{x^3}{6}+x\right|_0^1=\left[\frac{1^3}{6}+1\right]-\left[\frac{0^3}{6}+0\right]$

$E=\frac{1}{6}+1=\frac{7}{6}=1.166667$

Error = Exact – Approximate

Error = 1.171875 – 1.166667

Error = 0.00508

---

c). $I=\int_1^3 \frac{1}{x}\, dx \qquad n=6$

Sol: Given $I=\int_1^3 \frac{1}{x}\, dx \qquad n=6$

Compare with $I=\int_a^b f(x)\, dx$ we get

$a=1, \quad b=3, \quad f(x)=\frac{1}{x}$

Now $\Delta x=\frac{b-a}{n}$ putting values

$\Delta x=\frac{3-1}{6}=\frac{2}{6}=\frac{1}{3}$

Put $i=0$ corresponding value of function

$x_o=a=1 \qquad f_o=f(x_o)=\frac{1}{1}$

$f_o=f(1)=1$

Put $i=1$ corresponding value of function

$x_1=x_o+1.\Delta x \qquad f_1=f(x_1)=\frac{1}{\frac{4}{3}}$

$x_1=1+1\left(\frac{1}{3}\right) \qquad f_1=f\left(\frac{4}{3}\right)=\frac{3}{4}$

$x_1=\frac{4}{3}$

Put $i=2$ corresponding value of function

$x_2=x_o+2.\Delta x \qquad f_2=f(x_2)=\frac{1}{\frac{5}{3}}$

$x_2=1+2\left(\frac{1}{3}\right) \qquad f_2=f\left(\frac{5}{3}\right)=\frac{3}{5}$

$x_2=\frac{5}{3}$

Put $i=3$ corresponding value of function

$x_3=x_o+3.\Delta x$

$x_3=1+3\left(\frac{1}{3}\right) \qquad f_3=f(x_3)=\frac{1}{2}$

$x_3=2$

Put $i=4$ corresponding value of function

$x_4=x_o+4.\Delta x \qquad f_4=f(x_4)=\frac{1}{\frac{7}{3}}$

$x_4=1+4\left(\frac{1}{3}\right) \qquad f_4=f\left(\frac{7}{3}\right)=\frac{3}{7}$

$x_4=\frac{7}{3}$

Put $i=5$ corresponding value of function

$x_5=x_o+5.\Delta x \qquad f_5=f(x_5)=\frac{1}{\frac{8}{3}}$

$x_5=1+5\left(\frac{1}{3}\right) \qquad f_5=f\left(\frac{8}{3}\right)=\frac{3}{8}$

$x_5=\frac{8}{3}$

Put $i=6$ corresponding value of function

$x_6=x_o+6.\Delta x$

$x_6=1+6\left(\frac{1}{3}\right) \qquad f_6=f(x_6)=\frac{1}{3}$

$x_6=3=b$

Trapezoidal rule for $n=6$

$$I\approx T_4=\frac{\Delta x}{2}\left[f_o+2(f_1+f_2+f_3+f_4+f_5)+f_6\right]$$

putting

$$I\approx\frac{\frac{1}{3}}{2}\left[1+2\left(\frac{3}{4}+\frac{3}{5}+\frac{1}{2}+\frac{3}{7}+\frac{3}{8}\right)+\frac{1}{3}\right]$$

$$I\approx\frac{1}{6}\left[\frac{2789}{420}\right]=\frac{2789}{2520}$$

$I\approx 1.106746$

$$Exact=\int_1^3\frac{1}{x}\,dx$$

$E=\left(\ln|x|\right)_1^3=\left[\ln 3-\ln 1\right]$

$E=\ln 3-0=1.098612$

Error = Exact – Approximate

Error = 1.106746-1.098612

Error = 0.008134

---

d). $I=\int_0^2\sqrt{1+x^2}\,dx \qquad n=6$

Sol: Given $I=\int_0^2\sqrt{1+x^2}\,dx \qquad n=6$

Compare with $I=\int_a^b f(x)\,dx$ we get

$a=0, \quad b=1, \quad f(x)=\sqrt{1+x^2}$

Now $\Delta x=\frac{b-a}{n}$ putting the values

$\Delta x=\frac{2-0}{6}=\frac{1}{3}$

Put $i=0$ corresponding value of function

$f_o=f(x_o)=\sqrt{1+(0)^2}$

$x_o=a=0 \qquad f_o=f(0)=\sqrt{1}$

$f_o=f(0)=1$

Put $i=1$ corresponding value of function

$x_1=x_o+1.\Delta x \qquad f_1=f(x_1)=\sqrt{1+\left(\frac{1}{3}\right)^2}$

$x_1=0+1\left(\frac{1}{3}\right) \qquad f_1=f\left(\frac{1}{3}\right)=\sqrt{1+\frac{1}{9}}$

$f_1=f\left(\frac{1}{3}\right)=\sqrt{\frac{10}{9}}$

$x_1=\frac{1}{3} \qquad f_1=f\left(\frac{1}{3}\right)=\frac{\sqrt{10}}{3}$

Put $i=2$ corresponding value of function

$x_2=x_o+2.\Delta x \qquad f_2=f(x_2)=\sqrt{1+\left(\frac{2}{3}\right)^2}$

$x_2=0+2\left(\frac{1}{3}\right) \qquad f_2=f\left(\frac{2}{3}\right)=\sqrt{1+\frac{4}{9}}$

$f_2=f\left(\frac{2}{3}\right)=\sqrt{\frac{13}{9}}$

$x_2=\frac{2}{3} \qquad f_2=f\left(\frac{2}{3}\right)=\frac{\sqrt{13}}{3}$

Put $i=3$ corresponding value of function

$x_3=x_o+3.\Delta x \qquad f_3=f(x_3)=\sqrt{1+(1)^2}$

$x_3=0+3\left(\frac{1}{3}\right)=1 \qquad f_3=f(1)=\sqrt{2}$

Put $i=4$ corresponding value of function

$x_4=x_o+4.\Delta x \qquad f_4=f(x_4)=\sqrt{1+\left(\frac{4}{3}\right)^2}$

$x_2=0+4\left(\frac{1}{3}\right) \qquad f_4=f\left(\frac{4}{3}\right)=\sqrt{1+\frac{16}{9}}$

$f_4=f\left(\frac{4}{3}\right)=\sqrt{\frac{25}{9}}$

$x_2=\frac{4}{3} \qquad f_4=f\left(\frac{4}{3}\right)=\frac{5}{3}$

Put $i=5$ corresponding value of function

$x_5 = x_o + 5.\Delta x$ $\qquad f_5 = f(x_5) = \sqrt{1+\left(\frac{5}{3}\right)^2}$

$x_5 = 0 + 5\left(\frac{1}{3}\right)$ $\qquad f_5 = f\left(\frac{5}{3}\right) = \sqrt{1+\frac{25}{9}}$

$f_5 = f\left(\frac{5}{3}\right) = \sqrt{\frac{34}{9}}$

$x_5 = \frac{5}{3}$ $\qquad f_5 = f\left(\frac{5}{3}\right) = \frac{\sqrt{34}}{3}$

Put $i = 6$ corresponding value of function

$x_6 = x_o + 6.\Delta x$ $\qquad f_6 = f(x_6) = \sqrt{1+(2)^2}$

$x_6 = 0 + 6\left(\frac{1}{3}\right)$ $\qquad f_6 = f(2) = \sqrt{1+4}$

$x_6 = 2 = b$ $\qquad f_6 = f(2) = \sqrt{5}$

Trapezoidal rule for $n = 6$

$$I \approx T_4 = \frac{\Delta x}{2}\left[f_o + 2\left(f_1 + f_2 + f_3 + f_4 + f_5\right) + f_6\right]$$

putting

$$I \approx \frac{\frac{1}{3}}{2}\left[1 + 2\left(\frac{\sqrt{10}}{3} + \frac{\sqrt{13}}{3} + \sqrt{2} + \frac{5}{3} + \frac{\sqrt{34}}{3}\right) + \sqrt{5}\right]$$

$$I \approx \frac{1}{6}[17.79702]$$

$$I \approx 2.96617$$

$$Exact = \int_0^2 \sqrt{1+x^2}\ dx$$

$$E = \sqrt{1+x^2}\int_1^0 1dx - \int_0^1 x\sqrt{1+x^2}\,dx$$

$$E = \left|\frac{x\sqrt{1+x^2}}{2} + \frac{1}{2}\ln\left|x + \sqrt{x^2+1}\right|\right|_0^2$$

$$E = \left|\frac{2\sqrt{1+2^2}}{2} + \frac{1}{2}\ln\left|2+\sqrt{2^2+1}\right|\right| - \left|\frac{0\sqrt{1+0^2}}{2} + \frac{1}{2}\ln\left|0+\sqrt{0^2+1}\right|\right|$$

$$E = \left|\sqrt{5} + \frac{1}{2}\ln\left|2+\sqrt{5}\right|\right| - \left|0 + \frac{1}{2}\ln 1\right|$$

$$E = 2.23606 + 0.72188$$

$$E = 2.95794$$

Error = Exact – Approximate
Error = 2.96617-2.95764
Error = 0.002293

---

e) $I = \int_0^1 \frac{1}{\sqrt{1+x^2}}\ dx \qquad n = 5$

Sol: Given $I = \int_0^1 \frac{1}{\sqrt{1+x^2}}\ dx \qquad n = 5$

Compare with $I = \int_a^b f(x)\ dx$ we get

$a = 0, \qquad b = 1, \qquad f(x) = \frac{1}{\sqrt{1+x^2}}$

Now $\Delta x = \frac{b-a}{n}$ putting the values

$\Delta x = \frac{1-0}{5} \quad \Rightarrow \quad \Delta x = \frac{1}{5} = 0.2$

Put $i = 0$ corresponding value of function

$f_o = f(x_0) = \frac{1}{\sqrt{1+(0)^2}}$

$x_o = a = 0$ $\qquad f_o = f(0) = \frac{1}{\sqrt{1+0}}$

$f_o = f(0) = 1$

Put $i = 1$ corresponding value of function

$x_1 = x_o + 1.\Delta x$ $\qquad f_1 = f(x_1) = \frac{1}{\sqrt{1+(0.2)^2}}$

$x_1 = 0 + 1(0.2)$ $\qquad f_1 = f(0.2) = \frac{1}{\sqrt{1+0.04}}$

$x_1 = 0.2$ $\qquad f_1 = f(0.2) = \frac{1}{\sqrt{1.04}}$

Put $i = 2$ corresponding value of function

$x_2 = x_o + 2.\Delta x$ $\qquad f_2 = f(x_2) = \frac{1}{\sqrt{1+(0.4)^2}}$

$x_2 = 0 + 2(0.2)$

$x_2 = 0 + 0.4$ $\qquad f_2 = f(0.4) = \frac{1}{\sqrt{1+0.16}}$

$x_2 = 0.4$ $\qquad f_2 = f(0.4) = \frac{1}{\sqrt{1.16}}$

Put $i = 3$ corresponding value of function

$x_3 = x_o + 3.\Delta x$ $\qquad f_3 = f(x_3) = \frac{1}{\sqrt{1+(0.6)^2}}$

$x_3 = 0 + 3(0.2)$ $\qquad f_3 = f(0.6) = \frac{1}{\sqrt{1+0.36}}$

$x_3 = 0.6$ $\qquad f_3 = f(0.6) = \frac{1}{\sqrt{1.36}}$

Put $i = 4$ Corresponding value of function

$x_4 = x_o + 4.\Delta x$ $\qquad f_4 = f(x_4) = \frac{1}{\sqrt{1+(0.8)^2}}$

$x_4 = 0 + 4(0.2)$ $\qquad f_4 = f(0.8) = \frac{1}{\sqrt{1+0.64}}$

$x_4 = 0.8$ $\qquad f_4 = f(0.8) = \frac{1}{\sqrt{1.64}}$

Put $i = 4$ corresponding value of function

$x_5 = x_o + 5.\Delta x$ $\qquad f_5 = f(x_5) = \frac{1}{\sqrt{1+(1)^2}}$

$x_5 = 0 + 5(0.2)$ $\qquad f_5 = f(1) = \frac{1}{\sqrt{1+1}}$

$x_5 = 1 = b$ $\qquad f_5 = f(1) = \frac{1}{\sqrt{2}}$

Trapezoidal rule for $n = 5$

$I \approx T_4 = \frac{\Delta x}{2}\left[f_o + 2\left(f_1 + f_2 + f_3 + f_4\right) + f_5\right]$ putting

$$I \approx \frac{0.2}{2}\left[1 + 2\left(\frac{1}{\sqrt{1.04}} + \frac{1}{\sqrt{1.16}} + \frac{1}{\sqrt{1.36}} + \frac{1}{\sqrt{1.64}}\right) + \frac{1}{\sqrt{2}}\right]$$

$$I \approx 0.1[8.801945]$$

$$I \approx 0.8801945$$

$$Exact = \int_0^1 \frac{1}{\sqrt{1+x^2}}\ dx$$

$$E = \left|\sinh^{-1}(x)\right|_0^1$$

$$E = \sinh^{-1}(1) - \sinh^{-1}(0) = 0.881373587$$

Error = Exact – Approximate
Error = 0.8813736-0.8801945
Error = 0.0011791

Q2. Use the Simpson's rule to approximate the value of each definite integral. Round the answer to the nearest hundredth and compare your result with the exact value of the definite integral:

a). $I = \int_2^4 x^2\ dx \qquad n = 3$

Sol: Given $I = \int_2^4 x^2\ dx \qquad n = 3$

Compare with $I = \int_a^b f(x)\ dx$ we get

$a = 2, \qquad b = 4, \qquad f(x) = x^2$

Now $\Delta x=\dfrac{b-a}{2n}$ putting the values

$$\Delta x=\frac{4-2}{2(3)}=\frac{2}{2(3)}=\frac{1}{3}$$

Put $i=0$ corresponding value of function

$x_o=a=2$ $\quad f_o=f(x_o)=2^2=4$

Put $i=1$ corresponding value of function

$x_1=x_o+1.\Delta x$

$x_1=2+1\left(\frac{1}{3}\right)$ $\quad f_1=f(x_1)=\left(\frac{7}{3}\right)^2$

$x_1=2+\frac{1}{3}$ $\quad f_1=f\left(\frac{7}{3}\right)=\frac{49}{9}$

$x_1=\frac{7}{3}$

Put $i=2$ corresponding value of function

$x_2=x_o+2.\Delta x$

$x_2=2+2\left(\frac{1}{3}\right)$ $\quad f_2=f(x_2)=\left(\frac{8}{3}\right)^2$

$x_2=2+\frac{2}{3}$ $\quad f_2=f\left(\frac{8}{3}\right)=\frac{64}{9}$

$x_2=\frac{8}{3}$

Put $i=3$ corresponding value of function

$x_3=x_o+3.\Delta x$

$x_3=2+3\left(\frac{1}{3}\right)$ $\quad f_3=f(x_3)=(3)^2$

$x_3=2+1$ $\quad f_3=f(3)=9$

$x_3=3$

Put $i=4$ corresponding value of function

$x_4=x_o+4.\Delta x$

$x_4=2+4\left(\frac{1}{3}\right)$ $\quad f_4=f(x_4)=\left(\frac{10}{3}\right)^2$

$x_4=2+\frac{4}{3}$ $\quad f_4=f\left(\frac{10}{3}\right)=\frac{100}{9}$

$x_4=\frac{10}{3}$

Put $i=5$ corresponding value of function

$x_5=x_o+5.\Delta x$

$x_5=2+5\left(\frac{1}{3}\right)$ $\quad f_5=f(x_5)=\left(\frac{11}{3}\right)^2$

$x_5=2+\frac{5}{3}$ $\quad f_5=f\left(\frac{11}{3}\right)=\frac{121}{9}$

$x_5=\frac{11}{3}$

Put $i=6$ corresponding value of function

$x_6=x_o+6.\Delta x$

$x_6=2+6\left(\frac{1}{3}\right)$ $\quad f_6=f(x_6)=(4)^2$

$x_6=2+2$ $\quad f_6=f(4)=16$

$x_6=4=b$

Simpson's rule for $n=3$

$$I=\int_2^4 x^2\,dx\approx S_{2(3)}=\frac{\Delta x}{3}\left[f_o+4(f_1+f_3+f_5)+2(f_2+f_4)+f_6\right]$$

$$I\approx\frac{\frac{1}{3}}{3}\left[4+4\left(\frac{49}{9}+9+\frac{121}{9}\right)+2\left(\frac{64}{9}+\frac{100}{9}\right)+16\right]$$

$$I\approx\frac{1}{9}[168]=18.666667$$

$$Exact=\int_2^4 x^2\,dx$$

$$E=\left|\frac{x^3}{3}\right|_2^4=\left[\frac{4^3}{3}-\frac{2^3}{3}\right]=\frac{168}{9}$$

$E=18.666667$

Error = Exact – Approximate

Error = 18.666667-18.666667

Error = 0

---

b). $I=\displaystyle\int_2^3\left(\frac{x^2}{3}-1\right)dx \qquad n=4$

Sol: Given $I=\displaystyle\int_2^3\left(\frac{x^2}{3}-1\right)dx \qquad n=4$

Compare with $I=\displaystyle\int_a^b f(x)\,dx$ we get

$a=2,\qquad b=3,\qquad f(x)=\dfrac{x^2}{3}-1$

Now $\Delta x=\dfrac{b-a}{2n}$ putting the values

$\Delta x=\dfrac{3-2}{2(4)} \qquad \Rightarrow \Delta x=\dfrac{1}{8}$

Put $i=0$ corresponding value of function

$x_o=a=2$

$f_o=f(x_o)=\dfrac{2^2}{3}-1$

$f_o=f(2)=\dfrac{4}{3}-1$

$f_o=f(2)=\dfrac{1}{3}$

Put $i=1$ corresponding value of function

$x_1=x_o+1.\Delta x$ $\quad f_1=f(x_1)=\frac{1}{3}\left(\frac{17}{8}\right)^2-1$

$x_1=2+1\left(\frac{1}{8}\right)$ $\quad f_1=f\left(\frac{17}{8}\right)=\frac{1}{3}\left(\frac{289}{64}\right)-1$

$x_1=2+\frac{1}{8}$ $\quad f_1=f\left(\frac{17}{8}\right)=\frac{97}{192}$

$x_1=\frac{17}{8}$

Put $i=2$ corresponding value of function

$x_2=x_o+2.\Delta x$ $\quad f_2=f(x_2)=\frac{1}{3}\left(\frac{9}{4}\right)^2-1$

$x_2=2+2\left(\frac{1}{8}\right)$ $\quad f_2=f\left(\frac{9}{4}\right)=\frac{1}{3}\left(\frac{81}{16}\right)-1$

$x_2=2+\frac{1}{4}$ $\quad f_2=f\left(\frac{9}{4}\right)=\frac{11}{16}$

$x_2=\frac{9}{4}$

Put $i=3$ Corresponding value of function

$x_3=x_o+3.\Delta x$ $\quad f_3=f(x_3)=\frac{1}{3}\left(\frac{19}{8}\right)^2-1$

$x_3=2+3\left(\frac{1}{8}\right)$ $\quad f_3=f\left(\frac{19}{8}\right)=\frac{1}{3}\left(\frac{361}{64}\right)-1$

$x_3=2+\frac{3}{8}$ $\quad f_3=f\left(\frac{19}{8}\right)=\frac{169}{192}$

$x_3=\frac{19}{8}$

Put $i=4$ corresponding value of function

$x_4=x_o+4.\Delta x$ $\quad f_4=f(x_4)=\frac{1}{3}\left(\frac{5}{2}\right)^2-1$

$x_4=2+4\left(\frac{1}{8}\right)$ $\quad f_4=f\left(\frac{5}{2}\right)=\frac{1}{3}\left(\frac{25}{4}\right)-1$

$x_4=2+\frac{1}{2}$ $\quad f_4=f\left(\frac{5}{2}\right)=\frac{13}{12}$

$x_4=\frac{5}{2}$

Put $i=5$ corresponding value of function

$x_5=x_o+5.\Delta x$ $\quad f_5=f(x_5)=\frac{1}{3}\left(\frac{21}{8}\right)^2-1$

$x_5=2+5\left(\frac{1}{8}\right)$ $\quad f_5=f\left(\frac{21}{8}\right)=\frac{1}{3}\left(\frac{441}{64}\right)-1$

$x_5=2+\frac{5}{8}$ $\quad f_5=f\left(\frac{21}{8}\right)=\frac{83}{64}$

$x_5=\frac{21}{8}$

Put $i=6$ corresponding value of function

$x_6 = x_o + 6.\Delta x$

$x_6 = 2 + 6\left(\frac{1}{8}\right)$

$x_6 = 2 + \frac{3}{4}$

$x_6 = \frac{11}{4}$

$f_6 = f\left(x_6\right) = \frac{1}{3}\left(\frac{11}{4}\right)^2 - 1$

$f_6 = f\left(\frac{11}{4}\right) = \frac{1}{3}\left(\frac{121}{16}\right) - 1$

$f_6 = f\left(\frac{11}{4}\right) = \frac{73}{48}$

Put $i = 7$ corresponding value of function

$x_7 = x_o + 7.\Delta x$

$x_7 = 2 + 7\left(\frac{1}{8}\right)$

$x_7 = 2 + \frac{7}{8}$

$x_7 = \frac{23}{8}$

$f_7 = f\left(x_7\right) = \frac{1}{3}\left(\frac{23}{8}\right)^2 - 1$

$f_7 = f\left(\frac{23}{8}\right) = \frac{1}{3}\left(\frac{529}{64}\right) - 1$

$f_7 = f\left(\frac{23}{8}\right) = \frac{337}{192}$

Put $i = 8$ corresponding value of function

$x_8 = x_o + 8.\Delta x$

$x_8 = 2 + 8\left(\frac{1}{8}\right)$

$x_8 = 2 + 1$

$x_8 = 3 = b$

$f_8 = f\left(x_8\right) = \frac{1}{3}(3)^2 - 1$

$f_8 = f(3) = \frac{1}{3}(9) - 1$

$f_8 = f(3) = 3 - 1 = 2$

Simpson's rule for $n = 4$

$$I = \int_2^3 \left(\frac{x^2}{3} - 1\right) dx \approx S_{2(4)} = \frac{\Delta x}{3}\left[f_o + 4\left(f_1 + f_3 + f_5 + f_7\right) + 2\left(f_2 + f_4 + f_6\right) + f_8\right]$$

$$I = \frac{\frac{1}{8}}{3}\left[\frac{1}{8} + 4\left(\frac{97}{192} + \frac{169}{192} + \frac{83}{64} + \frac{337}{192}\right) + 2\left(\frac{11}{16} + \frac{13}{12} + \frac{73}{48}\right) + 2\right]$$

$$I = \int_2^3 \left(\frac{x^2}{3} - 1\right) dx \approx S_{2(4)} = \frac{1}{24}\left[\frac{635}{24}\right] = \frac{635}{576}$$

$I = 1.102431$

$$Exact = \int_2^3 \left(\frac{x^2}{3} - 1\right) dx$$

$$E = \left|\frac{x^3}{9} - x\right|_2^3 = \left[\frac{3^3}{9} - 3\right] - \left[\frac{2^3}{9} - 2\right]$$

$$E = \frac{10}{9} = 1.111111$$

Error = Exact – Approximate

Error = 1.111111-1.102431

Error = 0.008680

---

c). $I = \int_1^3 \frac{1}{x}\, dx \qquad n = 3$

Sol: Given $I = \int_1^3 \frac{1}{x}\, dx \qquad n = 3$

Compare with $I = \int_a^b f(x)\, dx$ we get

$a = 1, \qquad b = 3, \qquad f(x) = \frac{1}{x}$

Now $\Delta x = \frac{b-a}{n}$ putting the values

$\Delta x = \frac{3-1}{2(3)} \qquad \Rightarrow \Delta x = \frac{2}{6} = \frac{1}{3}$

Put $i = 0$ corresponding value of function

$x_o = a = 1$

$f_o = f\left(x_o\right) = \frac{1}{1}$

$f_o = f(1) = 1$

Put $i = 1$ corresponding value of function

$x_1 = x_o + 1.\Delta x$

$x_1 = 1 + 1\left(\frac{1}{3}\right)$

$x_1 = \frac{4}{3}$

$f_1 = f\left(x_1\right) = \frac{1}{\frac{4}{3}}$

$f_1 = f\left(\frac{4}{3}\right) = \frac{3}{4}$

Put $i = 2$ corresponding value of function

$x_2 = x_o + 2.\Delta x$

$x_2 = 1 + 2\left(\frac{1}{3}\right)$

$x_2 = \frac{5}{3}$

$f_2 = f\left(x_2\right) = \frac{1}{\frac{5}{3}}$

$f_2 = f\left(\frac{5}{3}\right) = \frac{3}{5}$

Put $i = 3$ corresponding value of function

$x_3 = x_o + 3.\Delta x$

$x_3 = 1 + 3\left(\frac{1}{3}\right)$

$x_3 = 2$

$f_3 = f\left(x_3\right) = \frac{1}{2}$

Put $i = 4$ corresponding value of function

$x_4 = x_o + 4.\Delta x$

$x_4 = 1 + 4\left(\frac{1}{3}\right)$

$x_4 = \frac{7}{3}$

$f_4 = f\left(x_4\right) = \frac{1}{\frac{7}{3}}$

$f_4 = f\left(\frac{7}{3}\right) = \frac{3}{7}$

Put $i = 5$ corresponding value of function

$x_5 = x_o + 5.\Delta x$

$x_5 = 1 + 5\left(\frac{1}{3}\right)$

$x_5 = \frac{8}{3}$

$f_5 = f\left(x_5\right) = \frac{1}{\frac{8}{3}}$

$f_5 = f\left(\frac{8}{3}\right) = \frac{3}{8}$

Put $i = 6$ corresponding value of function

$x_6 = x_o + 6.\Delta x$

$x_6 = 1 + 6\left(\frac{1}{3}\right)$

$x_6 = 3 = b$

$f_6 = f\left(x_6\right) = \frac{1}{3}$

Simpson's rule for $n = 3$

$$I = \int_1^3 \frac{1}{x}\, dx \approx S_{2(3)} = \frac{\Delta x}{3}\left[f_o + 4\left(f_1 + f_3 + + f_5\right) + 2\left(f_2 + f_4\right) + f_6\right]$$

$$I \approx S_6 = \frac{\frac{1}{3}}{3}\left[1 + 4\left(\frac{3}{4} + \frac{1}{2} + \frac{3}{8}\right) + 2\left(\frac{3}{5} + \frac{3}{7}\right) + \frac{1}{3}\right]$$

$$I \approx S_6 = \frac{1}{9}\left[\frac{2077}{210}\right] = \frac{2077}{1890}$$

$I \approx 1.098942$

$$Exact = \int_1^3 \frac{1}{x}\, dx = \left|\ln(x)\right|_1^3$$

$E = \ln(3) - \ln(1) = 1.098612$

Error = Exact – Approximate

Error = 1.098612-1.098942
Error = 0.000330

---

d). $I=\int_0^1 e^{2x}\ dx \qquad n=4$

Sol: Given $I=\int_0^1 e^{2x}\ dx \qquad n=4$

Compare with $I=\int_a^b f(x)\ dx$ we get

$a=0, \qquad b=1, \qquad f(x)=e^{2x}$

Now $\Delta x=\dfrac{b-a}{2n}$ putting the values

$\Delta x=\dfrac{1-0}{2(4)} \qquad \Rightarrow \qquad \Delta x=\dfrac{1}{8}$

Put $i=0$ corresponding value of function

$x_o=a=0$

$f_o=f(x_0)=e^{2(0)}$
$f_o=f(0)=e^0$
$f_o=f(0)=1$

Put $i=1$ Corresponding value of function

$x_1=x_o+1.\Delta x$
$x_1=0+1\left(\frac{1}{8}\right)$
$x_1=\frac{1}{8}$

$f_1=f(x_1)=e^{2\left(\frac{1}{8}\right)}$
$f_1=f\left(\frac{1}{8}\right)=e^{\frac{1}{4}}$

Put $i=2$ Corresponding value of function

$x_2=x_o+2.\Delta x$
$x_2=0+2\left(\frac{1}{8}\right)$
$x_2=\frac{1}{4}$

$f_2=f(x_2)=e^{2\left(\frac{1}{4}\right)}$
$f_2=f\left(\frac{1}{4}\right)=e^{\frac{1}{2}}$

Put $i=3$ corresponding value of function

$x_3=x_o+3.\Delta x$
$x_3=0+3\left(\frac{1}{8}\right)$
$x_3=\frac{3}{8}$

$f_3=f(x_3)=e^{2\left(\frac{3}{8}\right)}$
$f_3=f\left(\frac{3}{8}\right)=e^{\frac{3}{4}}$

Put $i=4$ corresponding value of function

$x_4=x_o+4.\Delta x$
$x_4=0+4\left(\frac{1}{8}\right)$
$x_4=\frac{1}{2}$

$f_4=f(x_4)=e^{2\left(\frac{1}{2}\right)}$
$f_4=f\left(\frac{1}{2}\right)=e^{1}$

Put $i=5$ corresponding value of function

$x_5=x_o+5.\Delta x$
$x_5=0+5\left(\frac{1}{8}\right)$
$x_5=\frac{5}{8}$

$f_5=f(x_5)=e^{2\left(\frac{5}{8}\right)}$
$f_5=f\left(\frac{5}{8}\right)=e^{\frac{5}{4}}$

Put $i=6$ corresponding value of function

$x_6=x_o+6.\Delta x$
$x_6=0+6\left(\frac{1}{8}\right)$
$x_6=\frac{3}{4}$

$f_6=f(x_6)=e^{2\left(\frac{3}{4}\right)}$
$f_6=f\left(\frac{3}{4}\right)=e^{\frac{3}{2}}$

Put $i=7$ Corresponding value of function

$x_7=x_o+7.\Delta x$
$x_7=0+7\left(\frac{1}{8}\right)$
$x_7=\frac{7}{8}$

$f_7=f(x_7)=e^{2\left(\frac{3}{4}\right)}$
$f_7=f\left(\frac{7}{8}\right)=e^{\frac{7}{4}}$

Put $i=8$ corresponding value of function

$x_8=x_o+8.\Delta x$
$x_8=0+8\left(\frac{1}{8}\right)$
$x_8=1$

$f_8=f(x_8)=e^{2(1)}$
$f_8=f(1)=e^2$

Simpson's rule for $n=4$

$$I=\int_0^1 e^{2x}\ dx\approx S_{2(4)}=\frac{\Delta x}{3}\Big[f_o+4\left(f_1+f_3+f_5+f_7\right)$$
$$+2\left(f_2+f_4+f_6\right)+f_8\Big]$$

$$I=\frac{\frac{1}{8}}{3}\left[1+4\left(e^{\frac{1}{4}}+e^{\frac{3}{4}}+e^{\frac{5}{4}}+e^{\frac{7}{4}}\right)+2\left(e^{\frac{1}{2}}+e^{1}+e^{\frac{3}{2}}\right)+e^2\right]$$

$$I=\int_0^1 e^{2x}\ dx\approx S_{2(4)}=\frac{1}{24}[76.670325]$$

$I=3.194597$

$$Exact=\int_0^1 e^{2x}\ dx$$

$$E=\left|\frac{e^{2x}}{2}\right|_0^1=\frac{1}{2}\left[e^2-e^0\right]$$

$E=3.194528$

Error = Exact – Approximate
Error = 3.194597 – 3.194528
Error = 0.000069

---

e). $I=\int_0^1 \dfrac{1}{2+x+x^2}\ dx \qquad n=2$

Sol: Given $I=\int_0^1 \dfrac{1}{2+x+x^2}\ dx \qquad n=2$

Compare with $I=\int_a^b f(x)\ dx$ we get

$a=0, \qquad b=1, \qquad f(x)=\dfrac{1}{2+x+x^2}$

Now $\Delta x=\dfrac{b-a}{2n}$ putting the values

$\Delta x=\dfrac{1-0}{2(2)} \qquad \Rightarrow \qquad \Delta x=\dfrac{1}{4}$

Put $i=0$ corresponding value of function

$x_o=a=0$

$f_o=f(x_0)=\dfrac{1}{2+(0)+(0)^2}$

$f_o=f(0)=\dfrac{1}{2}$

Put $i=1$ corresponding value of function

$x_1=x_o+1.\Delta x$
$x_1=0+1\left(\frac{1}{4}\right)$
$x_1=\frac{1}{4}$

$f_1=f(x_1)=\dfrac{1}{2+\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2}$

$f_1=f\left(\frac{1}{4}\right)=\dfrac{1}{2+\frac{1}{4}+\frac{1}{16}}$

$f_1=\dfrac{1}{\frac{32+4+1}{16}}=\dfrac{1}{\frac{37}{16}}=\dfrac{16}{37}$

Put $i=2$ corresponding value of function

$x_2=x_o+2.\Delta x$
$x_2=0+2\left(\frac{1}{4}\right)$
$x_2=\frac{1}{2}$

$f_2=f(x_2)=\dfrac{1}{2+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2}$

$f_2=f\left(\frac{1}{2}\right)=\dfrac{1}{2+\frac{1}{2}+\frac{1}{4}}$

$f_2=\dfrac{1}{\frac{8+2+1}{4}}=\dfrac{1}{\frac{11}{4}}=\dfrac{4}{11}$

Put $i=3$ corresponding value of function

$$x_3 = x_o + 3.\Delta x \qquad f_3 = f(x_3) = \frac{1}{2+\left(\frac{3}{4}\right)+\left(\frac{3}{4}\right)^2}$$

$$x_3 = 0 + 3\left(\tfrac{1}{4}\right) \qquad f_3 = f\left(\tfrac{3}{4}\right) = \frac{1}{2+\frac{3}{4}+\frac{9}{16}}$$

$$x_3 = \tfrac{3}{4} \qquad f_3 = \frac{1}{\frac{32+12+9}{16}} = \frac{1}{\frac{53}{16}} = \frac{16}{53}$$

Put $i = 4$ corresponding value of function

$$x_4 = x_o + 4.\Delta x \qquad f_4 = f(x_4) = \frac{1}{2+(1)+(1)^2}$$

$$x_4 = 0 + 4\left(\tfrac{1}{4}\right) \qquad f_4 = f(1) = \frac{1}{2+1+1}$$

$$x_4 = 1 \qquad f_4 = \frac{1}{4}$$

Simpson's rule for $n = 2$

$$I = \int_0^1 \frac{1}{2+x+x^2}\,dx \approx S_{2(2)}$$

$$I \approx \frac{\Delta x}{3}\left[f_o + 4(f_1+f_3) + 2(f_2) + f_4\right]$$

$$I = \frac{\frac{1}{4}}{3}\left[\frac{1}{2} + 4\left(\frac{16}{37}+\frac{16}{53}\right) + 2\left(\frac{4}{11}\right) + \frac{1}{4}\right]$$

$$I = \frac{1}{12}[4.414549627]$$

$$I = 0.367879135$$

$$Exact = \int_0^1 \frac{1}{x^2+x+2}\,dx$$

$$E = \int_0^1 \frac{1}{(x)^2 + 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + 2 - \left(\frac{1}{2}\right)^2}\,dx$$

$$E = \int_0^1 \frac{1}{\left(x+\frac{1}{2}\right)^2 + 2 - \frac{1}{4}}\,dx$$

$$E = \int_0^1 \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{7}{4}}\,dx$$

$$E = \int_0^1 \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}\,dx$$

$$E = \frac{2}{\sqrt{7}}\left|\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)\right|_0^1$$

$$E = \frac{2}{\sqrt{7}}\left[\tan^{-1}\left(\frac{1+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right) - \tan^{-1}\left(\frac{0+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)\right]$$

$$E = \frac{2}{\sqrt{7}}\left[\tan^{-1}\left(\frac{\frac{3}{2}}{\frac{\sqrt{7}}{2}}\right) - \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)\right]$$

$$E = \frac{2}{\sqrt{7}}\left[\tan^{-1}\left(\frac{3}{\sqrt{7}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{7}}\right)\right]$$

$$E = 0.3679068$$

Error = Exact – Approximate

Error = 0.3679068 – 0.36787913

Error = 0.0000155

Q3. A quarter circle of radius 1 has the equation $y = \sqrt{1-x^2}$ for $0 \le x \le 1$, which mean that $\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$, $n = 4$ approximate the definite integral on the left by trapezoidal rule that equals to the right side when $\pi = 3.1$

Sol: Given $\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$, $\quad n = 4$

Compare with $I = \int_a^b f(x)\,dx$ we get

$$a = 0, \qquad b = 1, \qquad f(x) = \sqrt{1-x^2}$$

Now $\Delta x = \frac{b-a}{n}$ putting the values

$$\Delta x = \frac{1-0}{4} \qquad \Rightarrow \qquad \Delta x = \frac{1}{4}$$

Put $i = 0$ corresponding value of function

$$x_o = a = 0 \qquad f_o = f(x_0) = \sqrt{1-(0)^2}$$

$$f_o = f(0) = \sqrt{1} = 1$$

Put $i = 1$ corresponding value of function

$$x_1 = x_o + 1.\Delta x \qquad f_1 = f(x_1) = \sqrt{1-\left(\tfrac{1}{4}\right)^2}$$

$$x_1 = 0 + 1\left(\tfrac{1}{4}\right) \qquad f_1 = f\left(\tfrac{1}{4}\right) = \sqrt{1-\tfrac{1}{16}}$$

$$x_1 = \tfrac{1}{4} \qquad f_1 = \sqrt{\tfrac{16-1}{16}} = \tfrac{\sqrt{15}}{4}$$

Put $i = 2$ corresponding value of function

$$x_2 = x_o + 2.\Delta x \qquad f_2 = f(x_2) = \sqrt{1-\left(\tfrac{1}{2}\right)^2}$$

$$x_2 = 0 + 2\left(\tfrac{1}{4}\right) \qquad f_2 = f\left(\tfrac{1}{2}\right) = \sqrt{1-\tfrac{1}{4}}$$

$$x_2 = \tfrac{1}{2} \qquad f_2 = \sqrt{\tfrac{4-1}{4}} = \tfrac{\sqrt{3}}{2}$$

Put $i = 3$ corresponding value of function

$$x_3 = x_o + 3.\Delta x \qquad f_3 = f(x_3) = \sqrt{1-\left(\tfrac{3}{4}\right)^2}$$

$$x_3 = 0 + 3\left(\tfrac{1}{4}\right) \qquad f_3 = f\left(\tfrac{3}{4}\right) = \sqrt{1-\tfrac{9}{16}}$$

$$x_3 = \tfrac{3}{4} \qquad f_3 = \sqrt{\tfrac{16-9}{16}} = \tfrac{\sqrt{7}}{4}$$

Put $i = 4$ corresponding value of function

$$x_4 = x_o + 4.\Delta x \qquad f_4 = f(x_4) = \sqrt{1-(1)^2}$$

$$x_4 = 0 + 4\left(\tfrac{1}{4}\right) \qquad f_4 = f(1) = \sqrt{1-1}$$

$$x_4 = 1 \qquad f_4 = \sqrt{0} = 0$$

Trapezoidal rule for $n = 4$

$I \approx T_4 = \frac{\Delta x}{2}\left[f_o + 2(f_1+f_2+f_3) + f_4\right]$ putting

$$I \approx \frac{\frac{1}{4}}{2}\left[1 + 2\left(\tfrac{\sqrt{15}}{4} + \tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{7}}{4}\right) + 0\right]$$

$$I \approx \frac{1}{8}[5.991418136]$$

$$I \approx 0.748927267 \simeq \frac{\pi}{4}$$

Q4. A quarter circle of radius 1 has the equation $y = \sqrt{1-x^2}$ for $0 \le x \le 1$, which mean that

$\int_0^1 \sqrt{1-x^2}\ dx = \frac{\pi}{4},\qquad n=4$ approximate the definite integral on the left by Simpson's rule that equals to the right side when $\pi = 3.1$

Sol: Given $\int_0^1 \sqrt{1-x^2}\ dx = \frac{\pi}{4},\qquad n=4$

Compare with $I = \int_a^b f(x)\ dx$ we get

$a=0,\qquad b=1,\quad f(x)=\sqrt{1-x^2}$

Now $\Delta x = \frac{b-a}{2n}$ putting the values

$\Delta x = \frac{1-0}{2(4)} \qquad \Rightarrow \qquad \Delta x = \frac{1}{8}$

Put $i=0$ corresponding value of function

$x_o = a = 0$ $\qquad f_o = f(x_0) = \sqrt{1-(0)^2}$

$f_o = f(0) = \sqrt{1} = 1$

Put $i=1$ corresponding value of function

$x_1 = x_o + 1.\Delta x$ $\qquad f_1 = f(x_1) = \sqrt{1-\left(\frac{1}{8}\right)^2}$

$x_1 = 0 + 1\left(\frac{1}{8}\right)$ $\qquad f_1 = f\left(\frac{1}{8}\right) = \sqrt{1-\frac{1}{64}}$

$x_1 = \frac{1}{8}$ $\qquad f_1 = \sqrt{\frac{64-1}{64}} = \frac{\sqrt{63}}{8}$

Put $i=2$ corresponding value of function

$x_2 = x_o + 2.\Delta x$ $\qquad f_2 = f(x_2) = \sqrt{1-\left(\frac{1}{4}\right)^2}$

$x_2 = 0 + 2\left(\frac{1}{8}\right)$ $\qquad f_2 = f\left(\frac{1}{4}\right) = \sqrt{1-\frac{1}{16}}$

$x_2 = \frac{1}{4}$ $\qquad f_2 = \sqrt{\frac{16-1}{16}} = \frac{\sqrt{15}}{4}$

Put $i=3$ corresponding value of function

$x_3 = x_o + 3.\Delta x$ $\qquad f_3 = f(x_3) = \sqrt{1-\left(\frac{3}{8}\right)^2}$

$x_3 = 0 + 3\left(\frac{1}{8}\right)$ $\qquad f_3 = f\left(\frac{3}{8}\right) = \sqrt{1-\frac{9}{64}}$

$x_3 = \frac{3}{8}$ $\qquad f_3 = \sqrt{\frac{64-9}{64}} = \frac{\sqrt{53}}{8}$

Put $i=4$ corresponding value of function

$x_4 = x_o + 4.\Delta x$ $\qquad f_4 = f(x_4) = \sqrt{1-\left(\frac{1}{2}\right)^2}$

$x_4 = 0 + 4\left(\frac{1}{8}\right)$ $\qquad f_4 = f\left(\frac{1}{2}\right) = \sqrt{1-\frac{1}{4}}$

$x_4 = \frac{1}{2}$ $\qquad f_4 = \sqrt{\frac{4-1}{4}} = \frac{\sqrt{3}}{2}$

Put $i=5$ corresponding value of function

$x_5 = x_o + 5.\Delta x$

$x_5 = 0 + 5\left(\frac{1}{8}\right)$

$x_5 = \frac{5}{8}$

$f_5 = f(x_5) = \sqrt{1-\left(\frac{5}{8}\right)^2}$

$f_5 = f\left(\frac{5}{8}\right) = \sqrt{1-\frac{25}{64}}$

$f_5 = \sqrt{\frac{64-25}{64}} = \frac{\sqrt{39}}{8}$

Put $i=6$ corresponding value of function

$x_6 = x_o + 6.\Delta x$

$x_6 = 0 + 6\left(\frac{1}{8}\right)$

$x_6 = \frac{3}{4}$

$f_6 = f(x_6) = \sqrt{1-\left(\frac{3}{4}\right)^2}$

$f_6 = f\left(\frac{3}{4}\right) = \sqrt{1-\frac{9}{16}}$

$f_6 = \sqrt{\frac{16-9}{16}} = \frac{\sqrt{7}}{4}$

Put $i=7$ corresponding value of function

$x_7 = x_o + 7.\Delta x$ $\qquad f_7 = f(x_7) = \sqrt{1-\left(\frac{7}{8}\right)^2}$

$x_7 = 0 + 7\left(\frac{1}{8}\right)$ $\qquad f_7 = f\left(\frac{7}{8}\right) = \sqrt{1-\frac{49}{64}}$

$x_7 = \frac{7}{8}$ $\qquad f_7 = \sqrt{\frac{64-49}{64}} = \frac{\sqrt{15}}{8}$

Put $i=8$ corresponding value of function

$x_8 = x_o + 8.\Delta x$ $\qquad f_8 = f(x_8) = \sqrt{1-(1)^2}$

$x_8 = 0 + 8\left(\frac{1}{8}\right)$ $\qquad f_8 = f(1) = \sqrt{1-1}$

$x_8 = 1$ $\qquad f_8 = \sqrt{0} = 0$

Simpson's rule for $n=4$

$$I = \int_0^1 \sqrt{1-x^2}\ dx \approx S_{2(4)} = \frac{\Delta x}{3}\left[f_o + 4(f_1+f_3+f_5+f_7) + 2(f_2+f_4+f_6) + f_8\right]$$

$$I = \frac{\frac{1}{8}}{3}\left[1 + 4\left(\frac{\sqrt{63}}{8}+\frac{\sqrt{53}}{8}+\frac{\sqrt{39}}{8}+\frac{\sqrt{15}}{8}\right) + 2\left(\frac{\sqrt{15}}{4}+\frac{\sqrt{3}}{2}+\frac{\sqrt{7}}{4}\right) + 0\right]$$

$$I = \int_0^1 \sqrt{1-x^2}\ dx \approx S_8 = \frac{1}{24}[18.65909]$$

$I = 0.777462113$

Q5. width of an irregularly shaped dam is measured at 5-m intervals, with result indicated in figure given below. Use Simpson's rule to estimate the area of the face of the dam

Hint:

$x: x_o = 0,\quad x_1 = 5\quad x_2 = 10\ x_3 = 15\quad x_4 = 20$

$f: f_o = 9,\quad f_1 = 15\ f_2 = 20\quad f_3 = 27\ f_4 = 30$

$\Delta x = 5,\qquad n = 2$

$$S_4 = \frac{\Delta x}{3}\left[f_o + 4(f_1+f_3) + 2f_2 + f_4\right]$$

Putting the values

$$S_4 = \frac{5}{3}\left[9 + 4(15+27) + 2(20) + 30\right]$$

$S_4 = 411.66m^2 \simeq 412m^2$