

Conjugate of a complex number.

For a complex number  $z = x + iy$ , the complex number  $\bar{z} = \overline{x + iy} = x - iy$  is called conjugate of  $z$ .

Here  $z$  and  $\bar{z}$  have same real part but their imaginary parts differ in sign, so  $z$  and  $\bar{z}$  are called conjugate of each other.

Thus  $5 + 4i$  and  $5 - 4i$  are conjugate complex numbers.

Note: If  $z = \bar{z}$ , then  $z$  is called self conjugate.

Since for  $a \in \mathbb{R}$

$$\bar{a} = a$$

$\Rightarrow$  Every real number is self conjugate.

Powers of  $i$

$$i^2 = -1 \quad \because i = \sqrt{-1}$$

$$i^3 = (i^2) \cdot i = (-1) \cdot i = -i$$

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

$$i^5 = (i^2)^2 \cdot i = (-1)^2 \cdot i = 1 \cdot i = i$$

$$i^6 = (i^2)^3 = (-1)^3 = -1$$

$$i^7 = (i^2)^3 \cdot i = (-1)^3 \cdot i = (-1) \cdot i = -i$$

$$i^8 = (i^2)^4 = (-1)^4 = 1$$

and so on.

$\Rightarrow$  Powers of  $i$  should be

expressed in terms of powers of  $i^2$ .

EXERCISE 1.2

① Verify the addition properties of complex numbers.

Solution:

i) Closure Property

For  $(a, b), (c, d) \in \mathbb{C}$

$$(a, b) + (c, d) = (a+c, b+d) \in \mathbb{C}$$

ii) Associative Property

For  $(a, b), (c, d), (e, f) \in \mathbb{C}$

$$((a, b) + (c, d)) + (e, f) = (a+c, b+d) + (e, f)$$

$$= ((a+c)+e, (b+d)+f)$$

$$= (a+(c+e), b+(d+f))$$

$\because$  '+' is associative in  $\mathbb{R}$

$$= (a, b) + (c+e, d+f)$$

$$= (a, b) + ((c, d) + (e, f))$$

iii) Additive Identity

$\forall (a, b) \in \mathbb{C}$  we have  $(0, 0) \in \mathbb{C}$  such that

$$(a, b) + (0, 0) = (a+0, b+0) = (a, b)$$

$\Rightarrow (0, 0)$  is additive identity in  $\mathbb{C}$

iv) Additive Inverse

$\forall (a, b) \in \mathbb{C}$  there is  $(-a, -b) \in \mathbb{C}$  such that

$$(a, b) + (-a, -b) = (a-a, b-b)$$

$$= (0, 0)$$

$\Rightarrow (a, b)$  and  $(-a, -b)$  are additive inverse of each other.

v) Commutative Property

$\forall (a, b), (c, d) \in \mathbb{C}$

The set  $C = \{x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}$  is called set of complex numbers.

### Operations on Complex Numbers:

Let  $z_1 = (x_1, y_1) = x_1 + iy_1$

$z_2 = (x_2, y_2) = x_2 + iy_2$

#### i) Addition:

$$\begin{aligned} z_1 + z_2 &= (x_1, y_1) + (x_2, y_2) \\ &= x_1 + iy_1 + x_2 + iy_2 \\ &= x_1 + x_2 + iy_1 + iy_2 \\ &= (x_1 + x_2) + i(y_1 + y_2) \\ &= (x_1 + x_2, y_1 + y_2) \end{aligned}$$

#### ii) Subtraction:

$$\begin{aligned} z_1 - z_2 &= (x_1, y_1) - (x_2, y_2) \\ &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= x_1 + iy_1 - x_2 - iy_2 \\ &= x_1 - x_2 + iy_1 - iy_2 \\ &= (x_1 - x_2) + i(y_1 - y_2) \\ &= (x_1 - x_2, y_1 - y_2) \end{aligned}$$

#### iii) Multiplication

$$\begin{aligned} z_1 z_2 &= (x_1, y_1) \cdot (x_2, y_2) \\ &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + ix_2 y_1 + i^2 y_1 y_2 \\ &= x_1 x_2 + ix_1 y_2 + ix_2 y_1 + (-1) y_1 y_2 \\ &= x_1 x_2 + ix_1 y_2 + ix_2 y_1 - y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \\ &= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \end{aligned}$$

#### iv) Division:

$$\frac{z_1}{z_2} = \frac{(x_1, y_1)}{(x_2, y_2)} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$\begin{aligned} &= \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} \\ &= \frac{x_1 x_2 - ix_1 y_2 + ix_2 y_1 - i^2 y_1 y_2}{x_2^2 - i^2 y_2^2} \\ &= \frac{x_1 x_2 - ix_1 y_2 + ix_2 y_1 - (-1) y_1 y_2}{x_2^2 - (-1) y_2^2} \\ &= \frac{x_1 x_2 - ix_1 y_2 + ix_2 y_1 + y_1 y_2}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 + y_1 y_2 + ix_2 y_1 - ix_1 y_2}{x_2^2 + y_2^2} \\ &= \frac{(x_1 x_2 + y_1 y_2)}{x_2^2 + y_2^2} + i \frac{(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \end{aligned}$$

$$\begin{aligned} &= \left( \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} \right) + i \left( \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \right) \\ &= \left( \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \right) \end{aligned}$$

#### v) Equality:

$(x_1, y_1) = (x_2, y_2) \Leftrightarrow x_1 = x_2, y_1 = y_2$

#### vi) Scalar Multiplication

If  $z = (x, y) = x + iy$  be a complex number and  $k$  is any real constant then

$$\begin{aligned} kz &= k(x, y) = k(x + iy) \\ &= kx + ik y \\ &= (kx, ky) \end{aligned}$$

Note that Every real number  $a$  can be written as  $a = a + i \cdot 0$   
 $\therefore$  Set of real numbers is subset of set of complex numbers.

$$(a, b) + (c, d) = (a+c, b+d) \quad \text{--- (12)}$$

$$= (c+a, d+b)$$

$\because '+'$  is commutative in  $\mathbb{R}$

$$= (c, d) + (a, b) \quad \times$$

② Verify the multiplication properties of the complex numbers.

Solution:

### i) Closure Property

$$\forall (a, b), (c, d) \in \mathbb{C}$$

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc) \in \mathbb{C}$$

$\therefore 'x'$  is closed in  $\mathbb{R}$

### ii) Associative Property

$$\text{For } (a, b), (c, d), (e, f) \in \mathbb{C}$$

Consider

$$\begin{aligned} [(a, b) \cdot (c, d)] \cdot (e, f) &= (ac - bd, ad + bc) \cdot (e, f) \\ &= ((ac - bd)e - (ad + bc)f, (ac - bd)f + (ad + bc)e) \\ &= (ace - bde - adf - bcf, acf - bdf + ade + bce) \end{aligned}$$

Now consider

$$\begin{aligned} (a, b) \cdot [(c, d) \cdot (e, f)] &= (a, b) \cdot [(ce - df), cf + de] \\ &= (a(ce - df) - b(cf + de), a(cf + de) + b(ce - df)) \\ &= (ace - adf - bcf - bde, acf + ade + bce - bdf) \end{aligned}$$

$\therefore$  From ① & ②, we get

$$[(a, b) \cdot (c, d)] \cdot (e, f) = (a, b) \cdot [(c, d) \cdot (e, f)]$$

### iii) Multiplicative Identity:

$$\forall (a, b) \in \mathbb{C} \text{ we have } (1, 0) \in \mathbb{C}$$

such that

$$\begin{aligned} (a, b) \cdot (1, 0) &= (a - 0, 0 + b) \\ &= (a, b) \end{aligned}$$

$\Rightarrow (1, 0)$  is multiplicative identity in  $\mathbb{C}$ .

### iv) Multiplicative Inverse.

$\forall (a, b) \in \mathbb{C}$  there is

$$\begin{aligned} \frac{1}{(a, b)} &= \frac{1}{a + ib} = \frac{1}{a + ib} \times \frac{a - ib}{a - ib} \\ &= \frac{a - ib}{a^2 - i^2 b^2} = \frac{a - ib}{a^2 - (-1)b^2} = \frac{a - ib}{a^2 + b^2} \\ &= \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \in \mathbb{C} \text{ such that} \end{aligned}$$

$$(a, b) \cdot \frac{1}{(a, b)} = 1 = 1 + i \cdot 0 = (1, 0)$$

$\Rightarrow (a, b)$  and  $\frac{1}{(a, b)}$  are

multiplicative inverse of each other.

### v) Commutative Property

$$\forall (a, b), (c, d) \in \mathbb{C}$$

$$\begin{aligned} (a, b) \cdot (c, d) &= (ac - bd, ad + bc) \\ &= (ca - db, da + cb) \\ &\because 'x' \text{ is commutative in } \mathbb{R} \\ &= (c, d) \cdot (a, b) \end{aligned}$$

③ Verify the distributive law of complex numbers.

$$(a, b) [(c, d) + (e, f)] = (a, b)(c, d) + (a, b)(e, f)$$

$$\begin{aligned} \text{L.H.S.} &= (a, b) [(c, d) + (e, f)] \\ &= (a, b)(c + e, d + f) \\ &= (a(c + e) - b(d + f), a(d + f) + b(c + e)) \\ &= (ac + ae - bd - bf, ad + af + bc + be) \end{aligned}$$

$$\begin{aligned} \text{R.H.S.} &= (a, b) \cdot (c, d) + (a, b) \cdot (e, f) \\ &= (ac - bd, ad + bc) + (ae - bf, af + be) \\ &= (ac - bd + ae - bf, ad + bc + af + be) \end{aligned}$$

$\therefore$  From ① & ②

$$\text{L.H.S.} = \text{R.H.S.}$$

④ Simplify the following

i)  $i^9 = (i^2)^4 \cdot i = (-1)^4 \cdot i = 1 \cdot i = i$  Ans.

ii)  $i^{14} = (i^2)^7 = (-1)^7 = -1$  Ans.

iii)  $(-i)^{19} = [(-1) \cdot (i)]^{19} = (-1)^{19} \cdot i^{19} = -i^{19}$   
 $= -(i^2)^9 \cdot i = -(-1)^9 \cdot i$   
 $= -(-1) \cdot i = i$  Ans.

iv)  $(-1)^{-\frac{21}{2}} = \frac{1}{(-1)^{\frac{21}{2}}} = \frac{1}{[(-1)^{\frac{1}{2}}]^{21}} = \frac{1}{i^{21}}$   
 $= \frac{1}{(i^2)^{10} \cdot i} = \frac{1}{(-1)^{10} \cdot i} = \frac{1}{1 \cdot i}$   
 $= \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1}$   
 $= -i$  Ans.

⑤ Write in terms of  $i$

i)  $\sqrt{-1} b = i b$  Ans.

ii)  $\sqrt{-5} = \sqrt{(-1)(5)} = \sqrt{-1} \sqrt{5} = i\sqrt{5}$  Ans.

iii)  $\sqrt{-\frac{16}{25}} = \sqrt{(-1) \cdot \frac{16}{25}} = \sqrt{-1} \sqrt{\frac{16}{25}} = i \frac{4}{5}$   
 $= \frac{4}{5} i$  Ans.

iv)  $\sqrt{\frac{1}{-4}}$  Method I  
 $= \frac{\sqrt{1}}{\sqrt{-4}} = \frac{1}{\sqrt{(-1)(4)}} = \frac{1}{\sqrt{-1} \sqrt{4}} = \frac{1}{i \cdot 2}$   
 $= \frac{1}{2i} = \frac{1}{2i} \times \frac{i}{i} = \frac{i}{2i^2} = \frac{i}{2(-1)} = \frac{i}{-2}$   
 $= -\frac{i}{2}$  Ans.

⑥  $(7, 9) + (3, -5) = (7+3, 9-5) = (10, 4)$  Ans.

⑦  $(8, -5) - (-7, 4) = (8+7, -5-4)$   
 $= (15, -9)$  Ans.

⑧  $(2, 6) \cdot (3, 7)$   
 $= (2 \cdot 3 - 6 \cdot 7, 2 \cdot 7 + 6 \cdot 3)$   
 $= (6 - 42, 14 + 18)$   
 $= (-36, 32)$  Ans.

⑨  $(5, -4) \cdot (-3, -2)$   
 $= ((5)(-3) - (-4)(-2), (5)(-2) + (-4)(-3))$   
 $= (-15 - 8, -10 + 12) = (-23, 2)$  Ans.

⑩  $(0, 3) \cdot (0, 5)$   
 $= ((0)(0) - (3)(5), (0)(5) + (3)(0))$   
 $= (0 - 15, 0 + 0) = (-15, 0)$  Ans.

⑪  $(2, 6) \div (3, 7) = \frac{(2, 6)}{(3, 7)}$   
 $= \frac{2+6i}{3+7i} = \frac{2+6i}{3+7i} \times \frac{3-7i}{3-7i}$   
 $= \frac{6-14i+18i-42i^2}{(3)^2 - (7i)^2} = \frac{6+4i-42(-1)}{9-49i^2}$   
 $= \frac{6+4i+42}{9-49(-1)} = \frac{48+4i}{9+49} = \frac{48+4i}{58}$   
 $= \frac{48}{58} + \frac{4}{58} i = \frac{24}{29} + \frac{2}{29} i$   
 $= \left(\frac{24}{29}, \frac{2}{29}\right)$  Ans.

⑫  $(5, -4) \div (-3, -8)$   
 $= \frac{(5, -4)}{(-3, -8)} = \frac{5-4i}{-3-8i} = \frac{5-4i}{-3-8i} \times \frac{-3+8i}{-3+8i}$   
 $= \frac{-15+40i+12i-32i^2}{(-3)^2 - (8i)^2}$   
 $= \frac{-15+52i-32(-1)}{9-64i^2} = \frac{-15+52i+32}{9+64}$   
 $= \frac{17+52i}{73} = \frac{17}{73} + \frac{52i}{73}$   
 $= \left(\frac{17}{73}, \frac{52}{73}\right)$  Ans.

⑬ Let the two conjugate complex numbers be

$Z = x + iy$  and  $\bar{Z} = x - iy$  where  $x, y \in \mathbb{R}$

Sum =  $Z + \bar{Z}$   
 $= x + iy + x - iy$   
 $= 2x \in \mathbb{R} \quad \because x \in \mathbb{R}$

$$\text{Product} = z \cdot \bar{z}$$

$$= (x+iy)(x-iy)$$

$$= x^2 - i^2 y^2$$

$$= x^2 - (-1)y^2 = x^2 + y^2 \in \mathbb{R}$$

$$\because x, y \in \mathbb{R}$$

$$(4) \text{ i) } (-4, 7)$$

$$\text{let } z = (-4, 7)$$

$$\text{Multiplicative inverse of } z = \frac{1}{z}$$

$$= \frac{1}{(-4, 7)} = \frac{1}{-4+7i} = \frac{1}{-4+7i} \times \frac{-4-7i}{-4-7i}$$

$$= \frac{-4-7i}{(-4)^2 - (7i)^2} = \frac{-4-7i}{16-49i^2}$$

$$= \frac{-4-7i}{16-49(-1)} = \frac{-4-7i}{16+49} = \frac{-4-7i}{65}$$

$$= -\frac{4}{65} - \frac{7}{65}i = \left(-\frac{4}{65}, -\frac{7}{65}\right)$$

Ans.

$$\text{ii) let } z = (\sqrt{2}, -\sqrt{5})$$

$$\text{Multiplicative inverse of } z = \frac{1}{z}$$

$$= \frac{1}{(\sqrt{2}, -\sqrt{5})} = \frac{1}{\sqrt{2}-\sqrt{5}i}$$

$$= \frac{1}{\sqrt{2}-\sqrt{5}i} \times \frac{\sqrt{2}+\sqrt{5}i}{\sqrt{2}+\sqrt{5}i} = \frac{\sqrt{2}+\sqrt{5}i}{(\sqrt{2})^2 - (\sqrt{5}i)^2}$$

$$= \frac{\sqrt{2}+\sqrt{5}i}{2-5i^2} = \frac{\sqrt{2}+\sqrt{5}i}{2-5(-1)} = \frac{\sqrt{2}+\sqrt{5}i}{2+5}$$

$$= \frac{\sqrt{2}+\sqrt{5}i}{7} = \frac{\sqrt{2}}{7} + \frac{\sqrt{5}}{7}i$$

$$= \left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7}\right) \text{ Ans.}$$

$$\text{iii) Let } z = (1, 0)$$

$$\text{Multiplicative inverse of } z = \frac{1}{z}$$

$$= \frac{1}{(1, 0)} = \frac{1}{1+i \cdot 0} = \frac{1}{1+0} = \frac{1}{1} = 1$$

$$= 1+i \cdot 0 = (1, 0) \text{ Ans.}$$

$$(15) \text{ i) } a^2 + 4b^2$$

$$= a^2 - (-1)4b^2 = a^2 - i^2 2^2 b^2$$

$$= (a)^2 - (i2b)^2$$

$$= (a+i2b)(a-i2b)$$

$$= (a+2bi)(a-2bi) \text{ Ans.}$$

$$\text{ii) } 9a^2 + 16b^2$$

$$= 9a^2 - (-1)16b^2$$

$$= (3a)^2 - i^2 4^2 b^2 = (3a)^2 - (i4b)^2$$

$$= (3a+i4b)(3a-i4b)$$

$$= (3a+4bi)(3a-4bi) \text{ Ans.}$$

$$\text{iii) } 3x^2 + 3y^2$$

$$= 3(x^2 + y^2)$$

$$= 3[x^2 - (-1)y^2]$$

$$= 3[(x)^2 - i^2 y^2] = 3[(x)^2 - (iy)^2]$$

$$= 3(x+iy)(x-iy) \text{ Ans.}$$

$$(16) \text{ i) } \frac{2-7i}{4+5i} = \frac{2-7i}{4+5i} \times \frac{4-5i}{4-5i}$$

$$= \frac{8-10i-28i+35i^2}{(4)^2 - (5i)^2}$$

$$= \frac{8-38i+35(-1)}{16-25i^2} = \frac{8-38i-35}{16+25}$$

$$= \frac{-27-38i}{41} = -\frac{27}{41} - \frac{38i}{41} \text{ Ans.}$$

$$\text{ii) } \frac{(-2+3i)^2}{1+i} = \frac{(-2)^2 + (3i)^2 + 2(-2)(3i)}{1+i}$$

$$= \frac{4+9i^2-12i}{1+i} = \frac{4-9-12i}{1+i} = \frac{-5-12i}{1+i}$$

$$= \frac{-5-12i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{-5+5i-12i+12i^2}{1^2 - i^2} = \frac{-5-7i-12}{1-(-1)}$$

$$= \frac{-17-7i}{1+1} = \frac{-17-7i}{2} = -\frac{17}{2} - \frac{7}{2}i \text{ Ans.}$$

$$\text{iii) } \frac{i}{1+i} = \frac{i}{1+i} \times \frac{1-i}{1-i} = \frac{i-i^2}{1-i^2}$$

$$= \frac{i-(-1)}{1-(-1)} = \frac{i+1}{2} = \frac{1+i}{2}$$

$$= \frac{1}{2} + \frac{i}{2} \text{ Ans.}$$

\*\*\*\*\*