**Definition:** A set of three mutually perpendicular axes having origin $O$ which are fixed in the rigid body and rotating with it and which are such that the product of inertia with respect to them are zero are called "principal axes of inertia" or simply "principal axes" of body at point $O$.

**Definition:** An axis is called "principal axis of inertia" or simply "principal axis" of a rigid body if directions of angular momentum $\mathbf{L}$ and angular velocity $\boldsymbol{\omega}$ are same, when rigid body is rotating about this axis.

**Theorem:** Above two definitions of principal axes are equivalent.

**Proof:** Suppose that for a rigid body we have three mutually concurrent and mutually perpendicular axes for which first definition holds. Choosing these axes as Cartesian coordinate axes, the inertia matrix with respect to this coordinate system is given by

$$[\,\mathbf{I}\,] = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

If rigid body rotates about $x-$ axis, then its angular velocity has the form $\boldsymbol{\omega}_x = \begin{pmatrix} \omega_{x1} \\ 0 \\ 0 \end{pmatrix}$

As we know that $[\mathbf{L}_x] = [\,\mathbf{I}\,][\boldsymbol{\omega}_x]$

$$\Rightarrow \begin{pmatrix} L_{x1} \\ L_{x2} \\ L_{x3} \end{pmatrix} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}\begin{pmatrix} \omega_{x1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_{11}\omega_{x1} \\ 0 \\ 0 \end{pmatrix} = I_{11}\begin{pmatrix} \omega_{x1} \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{L}_x = I_{11}\boldsymbol{\omega}_x$$

This shows that angular momentum is parallel to angular velocity. Similarly, we can show that when body rotates about $y$ or $z$ axis then angular momentum is parallel to angular velocity. Hence second definition also holds for given axes.

Conversely, suppose that for a rigid body we have three mutually concurrent and mutually perpendicular axes for which second definition holds. Choosing these axes as Cartesian coordinate axes, and assuming that body rotates about $x-$ axis, we have, by supposition, angular momentum and angular velocity are parallel

$$\Rightarrow \quad \mathbf{L}_x = \lambda_1\boldsymbol{\omega}_x, \qquad \text{where } \lambda \text{ is constant}$$

$$\Rightarrow \quad L_{x1}\mathbf{i} + L_{x2}\mathbf{j} + L_{x3}\mathbf{k} = \lambda_1(\omega_{x1}\mathbf{i} + 0\mathbf{j} + 0\mathbf{k})$$

$$\Rightarrow \begin{pmatrix} L_{x1} \\ L_{x2} \\ L_{x3} \end{pmatrix} = \begin{pmatrix} \lambda_1\omega_{x1} \\ 0 \\ 0 \end{pmatrix} ------\rightarrow (1)$$

As we know that $[\mathbf{L}_x] = [\,\mathbf{I}\,][\boldsymbol{\omega}_x]$

$$\Rightarrow \begin{pmatrix} L_{x1} \\ L_{x2} \\ L_{x3} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}\begin{pmatrix} \omega_{x1} \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} I_{11}\omega_{x1} \\ I_{12}\omega_{x1} \\ I_{13}\omega_{x1} \end{pmatrix} ----\rightarrow (2)$$

From (1) and (2), we have

$$\begin{pmatrix} \lambda_1\omega_{x1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_{11}\omega_{x1} \\ I_{12}\omega_{x1} \\ I_{13}\omega_{x1} \end{pmatrix}$$

$$\Rightarrow \quad I_{12} = I_{13} = 0 \qquad \because \quad \omega_{x1} \neq 0$$

Similarly, assuming the rotation of body about $y-$ axis ($\mathbf{L}_y = \lambda_2\boldsymbol{\omega}_y$), we get, $I_{12} = I_{23} = 0$.

$\Rightarrow$ All product of inertia are zero. Hence first definition also holds for given axes. (Note: $\lambda_i = I_{ii}$, $i = 1,2,3$)

**Definition:** The moment of inertia with respect to a principal axis is called "principal moment of inertia".

**Theorem:** Prove that for a rigid body a set of three mutually perpendicular principal axes exists at given point.

**Proof:** As we know from the definition of principal axis that if a rigid body rotates bout principal axes, passing through a point $O$, then the angular momentum $\mathbf{L}$ and the angular velocity $\boldsymbol{\omega}$ of the body are in same direction. So we can write, $\mathbf{L} = \lambda\boldsymbol{\omega}$, where, $\lambda$ is constant

Let, $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$, $\quad \boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$

Then, $L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k} = \lambda(\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k})$

Comparing corresponding components on both sides of above vector equation, we get

$$L_1 = \lambda\omega_1, \quad L_2 = \lambda\omega_2, \quad L_3 = \lambda\omega_3 --------\rightarrow (1)$$

As we know that, $[\mathbf{L}] = [\,\mathbf{I}\,][\boldsymbol{\omega}]$

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} --------\rightarrow (2)$$

From (1) and (2), we get,

$$I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = \lambda\omega_1$$
$$I_{12}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 = \lambda\omega_2$$
$$I_{13}\omega_1 + I_{23}\omega_2 + I_{33}\omega_3 = \lambda\omega_3$$

This system can be written as,

$$\left.\begin{aligned} (I_{11}-\lambda)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = 0 \\ I_{12}\omega_1 + (I_{22}-\lambda)\omega_2 + I_{23}\omega_3 = 0 \\ I_{13}\omega_1 + I_{23}\omega_2 + (I_{33}-\lambda)\omega_3 = 0 \end{aligned}\right\} --------\rightarrow (3)$$

This is homogeneous system of three equations in three unknowns $\omega_1$, $\omega_2$ and $\omega_3$. This system will have non trivial solution if an only if

$$\begin{vmatrix} I_{11}-\lambda & I_{12} & I_{13} \\ I_{12} & I_{22}-\lambda & I_{23} \\ I_{13} & I_{23} & I_{33}-\lambda \end{vmatrix} = 0$$

This is cubic equation in $I$ which is called characteristic equation of inertia matrix $[\,\mathbf{I}\,]$. It has three roots, say, $\lambda_1$, $\lambda_2$ and $\lambda_3$, which are, in fact, principal moments of inertia. By substituting $\lambda = \lambda_1$ in system (3), we can obtain the ratios $\omega_1 : \omega_2 : \omega_3$, which give direction of principal axes relative to which moment of inertia is $\lambda_1$. Similarly, we can find direction of other two principal axes corresponding to moments of inertia $\lambda_2$ and $\lambda_3$. We can always find three mutually perpendicular principal axes because $[\,\mathbf{I}\,]$ is symmetric. This shows that there exists three mutually perpendicular principal axes passing through given point $O$.

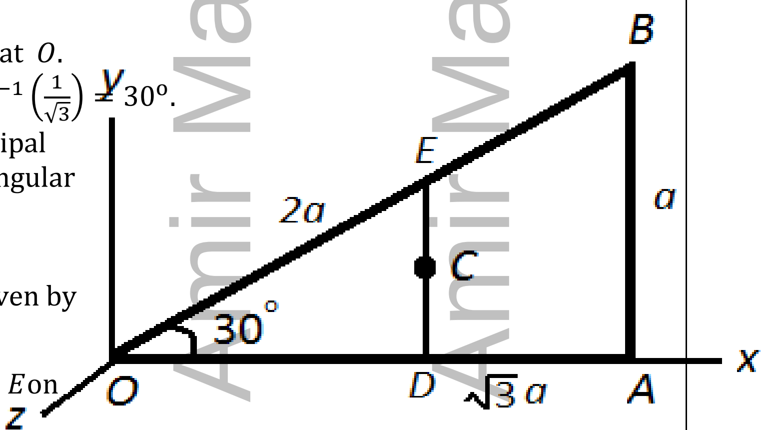
**Problem:** A triangular plate is made of uniform material and has sides of lengths $a$, $2a$ and $\sqrt{3}a$. Determine the (direction of) principal axes and corresponding principal moments of inertia at $30^{\text{o}}$ corner (or vertex).

**Solution:** Let $M$ and $\sigma$, respectively, be the mass and surface (areal) mass density of triangular plate $OAB$ lying in $xy$-plane, as shown in the figure, with $|OA| = \sqrt{3}a$, $|AB| = a$ and $|OB| = 2a$.

Clearly, $\quad |OB|^2 = (2a)^2 = \left(\sqrt{3}a\right)^2 + a^2 = |OA|^2 + |AB|^2$.

This shows that $OAB$ is right angled triangle with right angle at $O$.

Furthermore, $\tan(m\,\angle AOB) = \frac{|AB|}{|OA|} = \frac{a}{\sqrt{3}a} \Rightarrow m\,\angle AOB = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^{\text{o}}$.




Thus, we have to find principal axes and corresponding principal moments of inertia at vertex $O$. The moment of inertia of triangular plate about side $OA$ ($x$-axis)is given by

$$I_{xx} = I_{11} = I_{OA} = \frac{1}{6}M|AB|^2 = \frac{1}{6}Ma^2$$

The moment of inertia of triangular plate about side $AB$ is given by

$$I_{AB} = \frac{1}{6}M|OA|^2 = \frac{1}{6}M\left(\sqrt{3}a\right)^2 = \frac{1}{2}Ma^2$$

Let $C$ be the centre of mass of the plate and take $D$ on $OB$ and $E$ on $OA$ such that $DE$ is passing through $C$ and parallel to $AB$.

Then moment of inertia of plate about $DE$ is given by (using parallel axis theorem), as follows:

$$I_{DE} = I_{AB} - M|AD|^2 = \frac{1}{2}Ma^2 - M|AD|^2 - - - - - - - \rightarrow (1)$$

From figure, $\quad |AD| = |OA| - |OD| = \sqrt{3}a\,-$(x-coordinate of centre of mass C) $= \sqrt{3}a - \frac{1}{3}(x_A + x_A + x_A)$

$$= \sqrt{3}a - \frac{1}{3}\left(0 + \sqrt{3}a + \sqrt{3}a\right) = \sqrt{3}a - \frac{2\sqrt{3}a}{3} = \frac{3\sqrt{3}a - 2\sqrt{3}a}{3} = \frac{\sqrt{3}a}{3} = \frac{a}{\sqrt{3}} - - \rightarrow (2)$$

Using (2) in (1), we get, $\quad I_{DE} = \frac{1}{2}Ma^2 - M\left(\frac{a}{\sqrt{3}}\right)^2 = \frac{1}{2}Ma^2 - \frac{1}{3}Ma^2 = \frac{3Ma^2 - 2Ma^2}{6} = \frac{1}{6}Ma^2$

Then moment of inertia of plate about $y$-axis is given by (using parallel axis theorem), as follows,

$$I_{yy} = I_{22} = I_{DE} + M|OD|^2 = \frac{1}{6}Ma^2 + M(x\text{-coordinate of centre of mass } C)^2 = \frac{1}{6}Ma^2 + M\left(\frac{0 + \sqrt{3}a + \sqrt{3}a}{3}\right)^2$$

$$= \frac{1}{6}Ma^2 + M\left(\frac{2\sqrt{3}a}{3}\right)^2 = \frac{1}{6}Ma^2 + \frac{4}{3}Ma^2 = \frac{Ma^2 + 8Ma^2}{6} = \frac{9}{6}Ma^2 = \frac{3}{2}Ma^2$$

Then moment of inertia of plate about $z$-axis is given by (using perpendicular axis theorem), as follows,

$$I_{zz} = I_{33} = I_{xx} + I_{yy} = \frac{1}{6}Ma^2 + \frac{3}{2}Ma^2 = \frac{Ma^2 + 9Ma^2}{6} = \frac{10}{6}Ma^2 = \frac{5}{3}Ma^2$$

$$I_{xy} = I_{12} = -\int xy\,\mathrm{d}m = -\sigma\int xy\,\mathrm{d}x\mathrm{d}y = -\sigma\int_{x=0}^{\sqrt{3}a}\left(\int_{y=0}^{\frac{x}{\sqrt{3}}} xy\,\mathrm{d}y\right)\mathrm{d}x = -\sigma\int_{x=0}^{\sqrt{3}a}\left(x\left(\frac{y^2}{2}\right)\Big|_{y=0}^{\frac{x}{\sqrt{3}}}\right)\mathrm{d}x \qquad \because \mathrm{d}m = \sigma\mathrm{d}x\mathrm{d}y$$

$$= -\frac{\sigma}{6}\int_{x=0}^{\sqrt{3}a} x^3\mathrm{d}x = -\frac{1}{6}\left(\frac{2M}{\sqrt{3}a^2}\right)\left(\frac{x^4}{4}\right)\Big|_{x=0}^{\sqrt{3}a} = -\frac{1}{6}\left(\frac{2M}{\sqrt{3}a^2}\right)\left(\frac{9a^4}{4}\right) = -\frac{\sqrt{3}}{4}Ma^2 \quad \because \sigma = \frac{M}{\frac{1}{2}|OA||AB|} = \frac{M}{\frac{1}{2}(\sqrt{3}a)(a)} = \frac{2M}{\sqrt{3}a^2}$$

As $z = 0$ in $xy$-plane, therefore, $\quad I_{xz} = I_{13} = -\int xz\,\mathrm{d}m = 0$ and $I_{yz} = I_{23} = -\int yz\,\mathrm{d}m = 0$

The inertia matrix at point $O$, with respect to coordinate system $Oxyz$, is given by

$$[\mathbf{I}_O] = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{6}Ma^2 & -\frac{\sqrt{3}}{4}Ma^2 & 0 \\ -\frac{\sqrt{3}}{2}Ma^2 & \frac{3}{2}Ma^2 & 0 \\ 0 & 0 & \frac{5}{3}Ma^2 \end{pmatrix} = \begin{pmatrix} 2\alpha & -3\sqrt{3}\alpha & 0 \\ -3\sqrt{3}\alpha & 18\alpha & 0 \\ 0 & 0 & 20\alpha \end{pmatrix}, \quad \text{where} \quad \alpha = \frac{1}{12}Ma^2$$

To find the eigenvalues, we have the characteristic equation $\det([\mathbf{I}_O]-\lambda[I_3])=0$, where $[I_3]$ is unit matrix of order 3.

$$\det([\mathbf{I}_O]-\lambda[I_3])=0 \quad\Rightarrow\quad \begin{vmatrix} 2\alpha-\lambda & -3\sqrt{3}\alpha & 0 \\ -3\sqrt{3}\alpha & 18\alpha-\lambda & 0 \\ 0 & 0 & 20\alpha-\lambda \end{vmatrix}=0$$

On expanding by third row, we get,

$$(20\alpha-\lambda)\left[(2\alpha-\lambda)(18\alpha-\lambda)-\left(-3\sqrt{3}\alpha\right)^2\right]=0 \quad\Rightarrow\quad (20\alpha-\lambda)[36\alpha^2-2\alpha\lambda-18\alpha\lambda+\lambda^2-27\alpha^2]=0$$

$$\Rightarrow (20\alpha-\lambda)[\lambda^2-20\alpha\lambda+9\alpha^2]=0$$

| | |
|---|---|
| Either $20\alpha-\lambda=0 \;\Rightarrow\; \lambda=20\alpha$ | or, $\lambda^2-20\alpha\lambda+9\alpha^2=0 \;\Rightarrow\; \lambda=\frac{20\alpha\pm\sqrt{(20\alpha)^2-4(1)(9\alpha^2)}}{2(1)}$ $\Rightarrow\; \lambda=\frac{20\alpha\pm\sqrt{400\alpha^2-36\alpha^2}}{2}=\frac{20\alpha\pm\sqrt{364\alpha^2}}{2}=\frac{20\alpha\pm 2\sqrt{91}\alpha}{2}$ $=\left(10\pm\sqrt{91}\right)\alpha$ |

Thus, $\lambda_1=20\alpha$, $\lambda_2=\left(10+\sqrt{91}\right)\alpha$, and $\lambda_3=\left(10-\sqrt{91}\right)\alpha$

These eigenvalues gives principal moments of inertia at point $O$. To find the direction of corresponding principal axes, we find eigenvectors corresponding to each eigenvalue.

<u>**For $\boldsymbol{\lambda_1=20\alpha}$**</u>: Let $X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ be the required eigenvector corresponding to eigenvalue $\lambda_1=20\alpha$, then

$$([\mathbf{I}_O]-\lambda_1[I_3])X=\mathbf{0}\Rightarrow\begin{pmatrix}2\alpha-20\alpha & -3\sqrt{3}\alpha & 0\\ -3\sqrt{3}\alpha & 18\alpha-20\alpha & 0\\ 0 & 0 & 20\alpha-20\alpha\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}-18\alpha & -3\sqrt{3}\alpha & 0\\ -3\sqrt{3}\alpha & -2\alpha & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow\quad\begin{cases}-18\alpha x_1-3\sqrt{3}\alpha x_2=0\\ -3\sqrt{3}\alpha x_1-2\alpha x_2=0\end{cases}\quad\Rightarrow\begin{cases}6x_1+\sqrt{3}x_2=0 \;------(3)\\ 3\sqrt{3}x_1+2x_2=0\;-----(4)\end{cases}$$

From Eq. (3), we have $x_1=\frac{\sqrt{3}}{6}x_2$ and putting it in (4), we get, $-3\sqrt{3}\left(\frac{\sqrt{3}}{6}x_2\right)-2x_2=0\Rightarrow-\frac{3}{2}x_2-2x_2=0\Rightarrow x_2=0$. Put $x_2=0$ in (3), we get, $x_1=0$

Thus, $X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\r\end{pmatrix}$, where, $r\in\mathbb{R}$, $r\neq 0$ $\quad\Rightarrow\quad$ For $r=1$, we get, $X=\begin{pmatrix}0\\0\\1\end{pmatrix}=0\mathbf{i}+0\mathbf{j}+\mathbf{k}=\mathbf{k}$

<u>**For $\boldsymbol{\lambda_2=\left(10+\sqrt{91}\right)\alpha}$**</u>: Let $Y=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$ be the required eigenvector corresponding to eigenvalue $\lambda_2=\left(10+\sqrt{91}\right)\alpha$, then

$$([\mathbf{I}_O]-\lambda_2[I_3])Y=\mathbf{0}\quad\Rightarrow\quad\begin{pmatrix}-\left(8+\sqrt{91}\right)\alpha & -3\sqrt{3}\alpha & 0\\ -3\sqrt{3}\alpha & \left(8-\sqrt{91}\right)\alpha & 0\\ 0 & 0 & \left(10-\sqrt{91}\right)\alpha\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow\begin{cases}-\left(8+\sqrt{91}\right)\alpha y_1-3\sqrt{3}\alpha y_2=0\\ -3\sqrt{3}\alpha y_1+\left(8-\sqrt{91}\right)\alpha y_2=0\\ \left(10-\sqrt{91}\right)\alpha y_3=0\end{cases}\quad\Rightarrow\begin{cases}\left(8+\sqrt{91}\right)y_1+3\sqrt{3}y_2=0\;------(5)\\ 3\sqrt{3}y_1-\left(8-\sqrt{91}\right)y_2=0\;------(6)\\ y_3=0\end{cases}$$

From Eq. (5), we have $\frac{y_1}{y_2}=\frac{-3\sqrt{3}}{8+\sqrt{91}}$ and from Eq. (6), we have $\frac{y_1}{y_2}=\frac{8-\sqrt{91}}{3\sqrt{3}}=\frac{8-\sqrt{91}}{3\sqrt{3}}\cdot\frac{8+\sqrt{91}}{8+\sqrt{91}}=\frac{-27}{3\sqrt{3}\left(8+\sqrt{91}\right)}=\frac{-3\sqrt{3}}{8+\sqrt{91}}$

Thus, Eq. (5) and Eq. (6) are mutually identical, therefore, last system of equations can be written as

$$\begin{cases}\left(8+\sqrt{91}\right)y_1+3\sqrt{3}y_2=0\\ y_3=0\end{cases}$$

Let, $y_2=s$, where, $s\in\mathbb{R}$, $s\neq0$ $\quad\Rightarrow\quad y_1=\frac{-3\sqrt{3}}{8+\sqrt{91}}s$

Therefore, $Y=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}\frac{-3\sqrt{3}}{8+\sqrt{91}}s\\ s\\ 0\end{pmatrix}\Rightarrow$ For $s=-\left(8+\sqrt{91}\right)$, we get, $Y=\begin{pmatrix}3\sqrt{3}\\ -\left(8+\sqrt{91}\right)\\ 0\end{pmatrix}=3\sqrt{3}\mathbf{i}-\left(8+\sqrt{91}\right)\mathbf{j}$

<u>**For $\boldsymbol{\lambda_3=\left(10-\sqrt{91}\right)\alpha}$**</u>: Let $Z=\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}$ be the required eigenvector corresponding to eigenvalue

$\lambda_3 = (10-\sqrt{91})\alpha$, then

$$([\mathbf{I}_O]-\lambda_2[I_3])Z=\mathbf{0} \quad \Rightarrow \quad \begin{pmatrix} -(8-\sqrt{91})\alpha & -3\sqrt{3}\alpha & 0 \\ -3\sqrt{3}\alpha & (8+\sqrt{91})\alpha & 0 \\ 0 & 0 & (10+\sqrt{91})\alpha \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -(8-\sqrt{91})\alpha z_1 - 3\sqrt{3}\alpha z_2 = 0 \\ -3\sqrt{3}\alpha z_1 + (8+\sqrt{91})\alpha z_2 = 0 \\ (10+\sqrt{91})\alpha z_3 = 0 \end{cases} \quad \Rightarrow \begin{cases} (8-\sqrt{91})z_1 + 3\sqrt{3}z_2 = 0 \quad ------(7) \\ 3\sqrt{3}z_1 - (8+\sqrt{91})z_2 = 0 \quad ------(8) \\ z_3 = 0 \end{cases}$$

From Eq. (7), we have $\frac{z_1}{z_2}=\frac{-3\sqrt{3}}{8-\sqrt{91}}$ and from Eq. (8), we have $\frac{z_1}{z_2}=\frac{8+\sqrt{91}}{3\sqrt{3}}=\frac{8+\sqrt{91}}{3\sqrt{3}}\cdot\frac{8-\sqrt{91}}{8-\sqrt{91}}=\frac{-27}{3\sqrt{3}(8-\sqrt{91})}=\frac{-3\sqrt{3}}{8-\sqrt{91}}$

Thus, Eq. (7) and Eq. (8) are mutually identical, therefore, last system of equations can be written as

$$\begin{cases} (8-\sqrt{91})z_1 + 3\sqrt{3}z_2 = 0 \\ z_3 = 0 \end{cases}$$

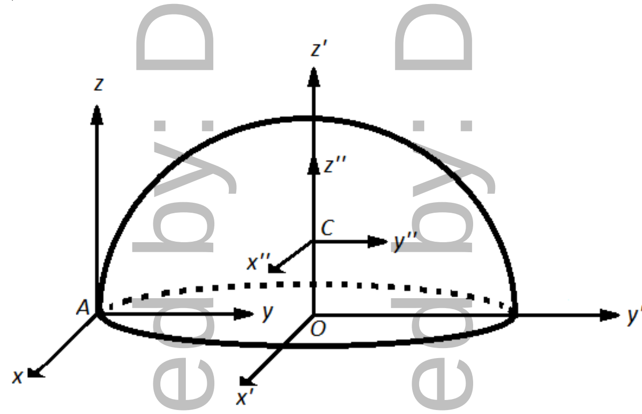
Let, $z_2 = t$, where, $t \in \mathbb{R}$, $t \neq 0$ $\quad\Rightarrow\quad z_1 = \frac{-3\sqrt{3}}{8-\sqrt{91}}t$

Therefore, $Z=\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}=\begin{pmatrix} \frac{-3\sqrt{3}}{8-\sqrt{91}}t \\ t \\ 0 \end{pmatrix} \Rightarrow$ For $t=-(8-\sqrt{91})$, we get, $Z=\begin{pmatrix} 3\sqrt{3} \\ -(8-\sqrt{91}) \\ 0 \end{pmatrix}=3\sqrt{3}\mathbf{i}-(8-\sqrt{91})\mathbf{j}$

| Principal moment of inertia | Principal axis | Normalized principal axis |
|---|---|---|
| $\lambda_1 = 20\alpha$ | $X=\mathbf{k}$ | $\hat{X}=\mathbf{i}$ |
| $\lambda_2 = (10+\sqrt{91})\alpha$ | $Y=3\sqrt{3}\mathbf{i}-(8+\sqrt{91})\mathbf{j}$ | $\hat{Y}=\frac{1}{\sqrt{182+16\sqrt{91}}}[3\sqrt{3}\mathbf{i}-(8+\sqrt{91})\mathbf{j}]$ |
| $\lambda_3 = (10-\sqrt{91})\alpha$ | $Z=3\sqrt{3}\mathbf{i}-(8-\sqrt{91})\mathbf{j}$ | $\hat{Z}=\frac{1}{\sqrt{182+16\sqrt{91}}}[3\sqrt{3}\mathbf{i}-(8-\sqrt{91})\mathbf{j}]$ |

**Problem:** Determine the (direction of) principal axes and corresponding principal moments of inertia of a uniform solid hemisphere at a point on its rim.

**Solution:** Let $M$, $a$ and $\rho$, respectively, be the mass, radius of the base and volume mass density of a uniform solid hemisphere. Let $A, O$ and $C$, respectively, be point on the rim, centre of the base and centre of mass of the hemisphere. Choose three coordinate axes $Axyz$, $Ox'y'z'$ and $Cx''y''z''$ as shown in the figure.




As we know that, the moments and product of inertia with respect to coordinate system $Ox'y'z'$ are given by $I_{O11}=I_{O22}=I_{O33}=\frac{2}{5}Ma^2$ and $I_{O12}=I_{O23}=I_{O13}=0$. Therefore, the inertia matrix with respect to coordinate system $Ox'y'z'$ is given by

$$[\mathbf{I}_O]=(I_{Oij})=\begin{pmatrix} I_{O11} & I_{O12} & I_{O13} \\ I_{O12} & I_{O22} & I_{O23} \\ I_{O13} & I_{O23} & I_{O33} \end{pmatrix}=\begin{pmatrix} \frac{2}{5}Ma^2 & 0 & 0 \\ 0 & \frac{2}{5}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 \end{pmatrix}$$

Next, we apply parallel axis theorem in tensor notation to find inertia tensor $[\mathbf{I}_C]$ with respect to coordinate system $Cx''y''z''$, as follows

$$I_{Oij}=I_{Cij}+M\mathbf{r}_c^2\delta_{ij}-Mx_{c,i}x_{c,j}$$
$$\Rightarrow \; I_{Cij}=I_{Oij}-M\mathbf{r}_c^2\delta_{ij}+Mx_{c,i}x_{c,j}$$

$$\Rightarrow \begin{pmatrix} I_{C11} & I_{C12} & I_{C13} \\ I_{C12} & I_{C22} & I_{C23} \\ I_{C13} & I_{C23} & I_{C33} \end{pmatrix}=\begin{pmatrix} I_{O11} & I_{O12} & I_{O13} \\ I_{O12} & I_{O22} & I_{O23} \\ I_{O13} & I_{O23} & I_{O33} \end{pmatrix}-M\begin{pmatrix} \mathbf{r}_c^2 & 0 & 0 \\ 0 & \mathbf{r}_c^2 & 0 \\ 0 & 0 & \mathbf{r}_c^2 \end{pmatrix}+M\begin{pmatrix} x_{c,1}x_{c,1} & x_{c,1}x_{c,2} & x_{c,1}x_{c,3} \\ x_{c,1}x_{c,2} & x_{c,2}x_{c,2} & x_{c,2}x_{c,3} \\ x_{c,1}x_{c,3} & x_{c,2}x_{c,3} & x_{c,3}x_{c,3} \end{pmatrix},$$

where, $\mathbf{r}_c=(x_{c,1},\ x_{c,2},\ x_{c,3})=\left(0,\ 0,\ \frac{3}{8}a\right)$ is the position vector of centre of mass $C$ with respect to coordinate system $Ox'y'z'$.

$$\Rightarrow \begin{pmatrix} I_{C11} & I_{C12} & I_{C13} \\ I_{C12} & I_{C22} & I_{C23} \\ I_{C13} & I_{C23} & I_{C33} \end{pmatrix} = \begin{pmatrix} \frac{2}{5}Ma^2 & 0 & 0 \\ 0 & \frac{2}{5}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 \end{pmatrix} - M\begin{pmatrix} \frac{9}{64}a^2 & 0 & 0 \\ 0 & \frac{9}{64}a^2 & 0 \\ 0 & 0 & \frac{9}{64}a^2 \end{pmatrix} + M\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{9}{64}a^2 \end{pmatrix}$$

$$\Rightarrow \quad [\mathbf{I}_C] = \begin{pmatrix} I_{C11} & I_{C12} & I_{C13} \\ I_{C12} & I_{C22} & I_{C23} \\ I_{C13} & I_{C23} & I_{C33} \end{pmatrix} = \begin{pmatrix} \frac{2}{5}Ma^2 - \frac{9}{64}Ma^2 & 0 & 0 \\ 0 & \frac{2}{5}Ma^2 - \frac{9}{64}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 - \frac{9}{64}Ma^2 + \frac{9}{64}Ma^2 \end{pmatrix} = \begin{pmatrix} \frac{83}{320}Ma^2 & 0 & 0 \\ 0 & \frac{83}{320}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 \end{pmatrix}.$$

Now, we apply parallel axis theorem in tensor notation to find inertia tensor $[\mathbf{I}_A]$ with respect to coordinate system $Axyz$, as follows

$$I_{Aij} = I_{Cij} + M\mathbf{r}_c'^2\delta_{ij} - Mx'_{c,i}x'_{c,j}$$

$$\Rightarrow \begin{pmatrix} I_{A11} & I_{A12} & I_{A13} \\ I_{A12} & I_{A22} & I_{A23} \\ I_{A13} & I_{A23} & I_{A33} \end{pmatrix} = \begin{pmatrix} I_{C11} & I_{C12} & I_{C13} \\ I_{C12} & I_{C22} & I_{C23} \\ I_{C13} & I_{C23} & I_{C33} \end{pmatrix} + M\begin{pmatrix} \mathbf{r}_c'^2 & 0 & 0 \\ 0 & \mathbf{r}_c'^2 & 0 \\ 0 & 0 & \mathbf{r}_c'^2 \end{pmatrix} - M\begin{pmatrix} x'_{c,1}x'_{c,1} & x'_{c,1}x'_{c,2} & x'_{c,1}x'_{c,3} \\ x'_{c,1}x'_{c,2} & x'_{c,2}x'_{c,2} & x'_{c,2}x'_{c,3} \\ x'_{c,1}x'_{c,3} & x'_{c,2}x'_{c,3} & x'_{c,3}x'_{c,3} \end{pmatrix},$$

where, $\mathbf{r}_c' = (x'_{c,1},\ x'_{c,1},\ x'_{c,1}) = \left(0,\ a,\ \frac{3}{8}a\right)$ is the position vector of centre of mass $C$ with respect to coordinate system $Axyz$.

$$\Rightarrow \begin{pmatrix} I_{A11} & I_{A12} & I_{A13} \\ I_{A12} & I_{A22} & I_{A23} \\ I_{A13} & I_{A23} & I_{A33} \end{pmatrix} = \begin{pmatrix} \frac{83}{320}Ma^2 & 0 & 0 \\ 0 & \frac{83}{320}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 \end{pmatrix} + M\begin{pmatrix} \frac{73}{64}a^2 & 0 & 0 \\ 0 & \frac{73}{64}a^2 & 0 \\ 0 & 0 & \frac{73}{64}a^2 \end{pmatrix} - M\begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & \frac{3}{8}a^2 \\ 0 & \frac{3}{8}a^2 & \frac{9}{64}a^2 \end{pmatrix}$$

$$[\mathbf{I}_A] = \begin{pmatrix} I_{A11} & I_{A12} & I_{A13} \\ I_{A12} & I_{A22} & I_{A23} \\ I_{A13} & I_{A23} & I_{A33} \end{pmatrix} = \begin{pmatrix} \frac{83}{320}Ma^2 + \frac{73}{64}Ma^2 & 0 & 0 \\ 0 & \frac{83}{320}Ma^2 + \frac{73}{64}Ma^2 - Ma^2 & -\frac{3}{8}Ma^2 \\ 0 & -\frac{3}{8}a^2 & \frac{2}{5}Ma^2 + \frac{73}{64}Ma^2 - \frac{9}{64}Ma^2 \end{pmatrix}$$

$$[\mathbf{I}_A] = \begin{pmatrix} \frac{7}{5}Ma^2 & 0 & 0 \\ 0 & \frac{2}{5}Ma^2 & -\frac{3}{8}Ma^2 \\ 0 & -\frac{3}{8}Ma^2 & \frac{7}{5}Ma^2 \end{pmatrix} = \begin{pmatrix} 56\alpha & 0 & 0 \\ 0 & 16\alpha & -15\alpha \\ 0 & -15\alpha & 56\alpha \end{pmatrix}, \quad \text{where,} \quad \alpha = \frac{1}{40}Ma^2$$

To find the eigenvalues, we have the characteristic equation $\det([I_A] - \lambda[I_3]) = 0$, where $[I_3]$ is unit matrix of order 3.

$$\det([\mathbf{I}_A] - \lambda[I_3]) = 0 \quad \Rightarrow \quad \begin{vmatrix} 56\alpha - \lambda & 0 & 0 \\ 0 & 16\alpha - \lambda & -15\alpha \\ 0 & -15\alpha & 56\alpha - \lambda \end{vmatrix} = 0$$

On expanding by first row, we get,

$(56\alpha - \lambda)[(16\alpha - \lambda)(56\alpha - \lambda) - (-15\alpha)^2] = 0 \quad \Rightarrow \quad (20\alpha - \lambda)[896\alpha^2 - 16\alpha\lambda - 56\alpha\lambda + \lambda^2 - 225\alpha^2] = 0$

$\Rightarrow (56\alpha - \lambda)[\lambda^2 - 72\alpha\lambda + 671\alpha^2] = 0$

| | |
|---|---|
| Either $\quad 56\alpha - \lambda = 0 \Rightarrow \lambda = 56\alpha$ | or, $\quad \lambda^2 - 72\alpha\lambda + 671\alpha^2 = 0 \Rightarrow \lambda = \frac{72\alpha \pm \sqrt{(72\alpha)^2 - 4(1)(671\alpha^2)}}{2(1)}$<br>$\Rightarrow \lambda = \frac{72\alpha \pm \sqrt{5184\alpha^2 - 2684\alpha^2}}{2} = \frac{72\alpha \pm \sqrt{2500\alpha^2}}{2} = \frac{72\alpha \pm 50\alpha}{2}$<br>$\Rightarrow \lambda = \frac{72\alpha + 50\alpha}{2}, \frac{72\alpha - 50\alpha}{2} = \frac{122\alpha}{2}, \frac{22\alpha}{2} = 61\alpha, 11\alpha$ |

Thus, $\quad \lambda_1 = 56\alpha, \quad \lambda_2 = 61\alpha, \quad$ and $\quad \lambda_3 = 11\alpha.$

These eigenvalues gives principal moments of inertia an $A$. To find the direction of corresponding principal axes, we find eigenvectors corresponding to each eigenvalue.

**For $\lambda_1 = 56\alpha$:** Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the required eigenvector corresponding to eigenvalue $\lambda_1 = 56\alpha$, then

$$([\mathbf{I}_A] - \lambda_1[I_3])X = \mathbf{0} \Rightarrow \begin{pmatrix} 56\alpha - 56\alpha & 0 & 0 \\ 0 & 16\alpha - 56\alpha & -15\alpha \\ 0 & -15\alpha & 56\alpha - 56\alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -40\alpha & -15\alpha \\ 0 & -15\alpha & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -40\alpha x_2 - 15\alpha x_3 = 0 \\ -15\alpha x_2 = 0 \end{cases} \Rightarrow \begin{cases} 8x_2 - 3x_3 = 0 \text{ ------- } (1) \\ x_2 = 0 \text{ ------ } (2) \end{cases}$$

Thus we have, $x_2 = x_3 = 0$ and $x_1 = r$, where, $r \in \mathbb{R}$, $r \neq 0$

Thus, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$, $\Rightarrow$ For $r = 1$, we get, $X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{i}$

**For $\lambda_2 = 61\alpha$:** Let $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ be the required eigenvector corresponding to eigenvalue $\lambda_2 = 61\alpha$, then

$$([\mathbf{I}_A] - \lambda_2[I_3])Y = \mathbf{0} \Rightarrow \begin{pmatrix} 56\alpha - 61\alpha & 0 & 0 \\ 0 & 16\alpha - 61\alpha & -15\alpha \\ 0 & -15\alpha & 56\alpha - 61\alpha \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -5\alpha & 0 & 0 \\ 0 & -45\alpha & -15\alpha \\ 0 & -15\alpha & -5\alpha \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -5\alpha y_1 = 0 \\ -45\alpha y_2 - 15\alpha y_3 = 0 \\ -15\alpha y_2 - 5\alpha y_3 = 0 \end{cases} \Rightarrow \begin{cases} y_1 = 0 \\ 3y_2 + y_3 = 0 \\ 3y_2 + y_3 = 0 \end{cases} \Rightarrow \begin{cases} y_1 = 0 \\ 3y_2 + y_3 = 0 \end{cases}$$

Let, $y_2 = s$, where, $s \in \mathbb{R}$, $s \neq 0$ $\Rightarrow$ $y_3 = -3s$

Thus, $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ s \\ -3s \end{pmatrix}$ $\Rightarrow$ For $s = 1$, we get, $Y = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = 0\mathbf{i} + \mathbf{j} - 3\mathbf{k} = \mathbf{j} - 3\mathbf{k}$

**For $\lambda_3 = 11\alpha$:** Let $Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ be the required eigenvector corresponding to eigenvalue $\lambda_3 = 11\alpha$, then

$$([\mathbf{I}_A] - \lambda_2[I_3])Z = \mathbf{0} \Rightarrow \begin{pmatrix} 56\alpha - 11\alpha & 0 & 0 \\ 0 & 16\alpha - 11\alpha & -15\alpha \\ 0 & -15\alpha & 56\alpha - 11\alpha \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 45\alpha & 0 & 0 \\ 0 & 5\alpha & -15\alpha \\ 0 & -15\alpha & 45\alpha \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -45\alpha z_1 = 0 \\ 5\alpha z_2 - 15\alpha z_3 = 0 \\ -15\alpha z_2 + 45\alpha z_3 = 0 \end{cases} \Rightarrow \begin{cases} z_1 = 0 \\ z_2 - 3z_3 = 0 \\ z_2 - 3z_3 = 0 \end{cases} \Rightarrow \begin{cases} z_1 = 0 \\ z_2 - 3z_3 = 0 \end{cases}$$

Let, $z_3 = t$, where, $t \in \mathbb{R}$, $t \neq 0$ $\Rightarrow$ $z_2 = 3t$

Thus, $Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3t \\ t \end{pmatrix}$ $\Rightarrow$ For $t = 1$, we get, $Z = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = 0\mathbf{i} + 3\mathbf{j} + \mathbf{k} = 3\mathbf{j} + \mathbf{k}$

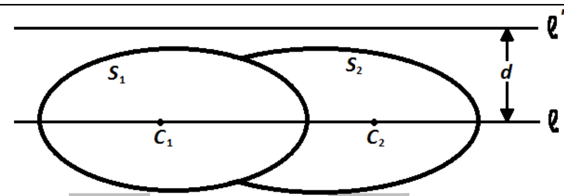
| Principal moment of inertia | Principal axis | Normalized principal axis |
|---|---|---|
| $\lambda_1 = 56\alpha$ | $X = \mathbf{i}$ | $\hat{X} = \mathbf{i}$ |
| $\lambda_2 = 61\alpha$ | $Y = \mathbf{j} - 3\mathbf{k}$ | $\hat{Y} = (1/\sqrt{10})(\mathbf{j} - 3\mathbf{k})$ |
| $\lambda_3 = 11\alpha$ | $Z = 3\mathbf{j} + \mathbf{k}$ | $\hat{Z} = (1/\sqrt{10})(3\mathbf{j} + \mathbf{k})$ |

**Definition:** Two distributions of matter are said to be "equimomental" if they have the same moment of inertia about any line in spase.

**Theorem:** Two systems $S_1$ and $S_2$ are equimomental if and only if the following three conditions are satisfied,
$(i)$ they have same mass,
$(ii)$ they have same centre of mass, and
$(iii)$ they have same principal axes and principal moments of inertia at centre of mass.

**Proof:** Suppose that two systems $S_1$ and $S_2$ are equimomental. We will show that conditions $(i)$, $(ii)$ and $(iii)$ are satisfied.

**$(i)$** Let $M_1$ and $M_2$, respectively, be the masses of the systems $S_1$ and $S_2$ and $C_1$ and $C_2$, respectively, be their centres of mass. Since the systems are supposed to be equimomental, therefore their moments of inertia about any line should be same. In particular, their moments of inertia about line $l$ through $C_1$ and $C_2$ should also be same, say, $I_l$. Let $l'$ be any line parallel to $l$ and $d$ be the perpendicular distance between $l$ and $l'$. Further suppose that $I_{l'}$ be the common moment of inertia of both systems about line $l'$.




By parallel axis theorem, we have,

$$I_{l'} = I_l + M_1 d^2 \quad \text{(for system } S_1) \;------\to (1)$$

$$I_{l'} = I_l + M_2 d^2 \quad \text{(for system } S_2) \;------\to (2)$$

From equations (1) and (2), we have,

$$I_l + M_1 d^2 = I_l + M_2 d^2 \quad \Rightarrow \quad M_1 = M_2 = M \text{ (say)}$$

$\Rightarrow$ masses of both systems are same $\Rightarrow$ condition $(i)$ is satisfied.

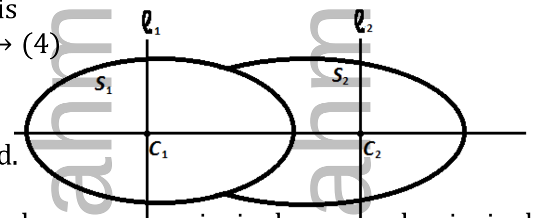
**$(ii)$** Now, let $l_1$ and $l_2$, respectively, be the lines through $C_1$ and $C_2$ and perpendicular to line $l$. Let common moment of inertia of each system about line $l_1$ be $I_{l_1}$ and about line $l_2$ be $I_{l_2}$.

By parallel axis theorem, moment of inertia of system $S_1$ about $l_2$ is

$$I_{l_2} = I_{l_1} + M|C_1C_2|^2 \;-------\to (3)$$

Again, by parallel axis theorem, moment of inertia of system $S_2$ about $l_2$ is

$$I_{l_2} = I_{l_1} - M|C_1C_2|^2 \;-------\to (4)$$




From equations (3) and (4), we get

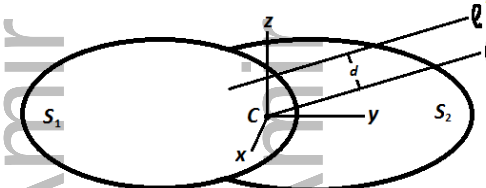
$I_{l_1} + M|C_1C_2|^2 = I_{l_1} - M|C_1C_2|^2 \;\Rightarrow\; |C_1C_2| = 0 \;\Rightarrow\; C_1 \equiv C_2 \equiv C$ (say)

$\Rightarrow$ centres of mass of both systems are same $\Rightarrow$ condition $(ii)$ is satisfied.

**$(iii)$** Since both system have same centre of mass $C$ and same mass $M$,

Therefore, they both have same momental ellipsoid at $C$. Hence, they have same principal axes and principal moments of inertia at centre of mass $C$. $\Rightarrow$ condition $(iii)$ is satisfied.

Conversely, suppose that for two systems $S_1$ and $S_2$, conditions $(i)$, $(ii)$ and $(iii)$ are satisfied. We will show that both systems are equimomental.




Let $C$ and $M$, respectively, be the common centre of mass and common mass of both systems. Further let that $I_1$, $I_2$ and $I_3$ be the common principal moments of inertia about common principal axes at centre of mass $C$. In figure, common principal axes at $C$ are shown by Cartesian coordinate system $Cxyz$.

Let $l$ be an arbitrary line in space. Draw a line $l'$ through $C$ parallel to $l$.

Then the moment of inertia of each system about $l'$ is given by

$$I_{l'} = I_1\lambda^2 + I_2\mu^2 + I_3\nu^2,$$

where, $\lambda$, $\mu$ and $\nu$ are direction cosines of line $l'$. Now, by using parallel axis theorem, the moment of inertia of each system about line $l$ is given by

$$I_l = I_{l'} + Md^2 = I_1\lambda^2 + I_2\mu^2 + I_3\nu^2 + Md^2,$$

where, $d$ is the perpendicular distance between lines $l$ and $l'$. Since the moment of inertia of both system about an arbitrary line $l$ in space is same. This shows that both systems $S_1$ and $S_2$ are equimomental.

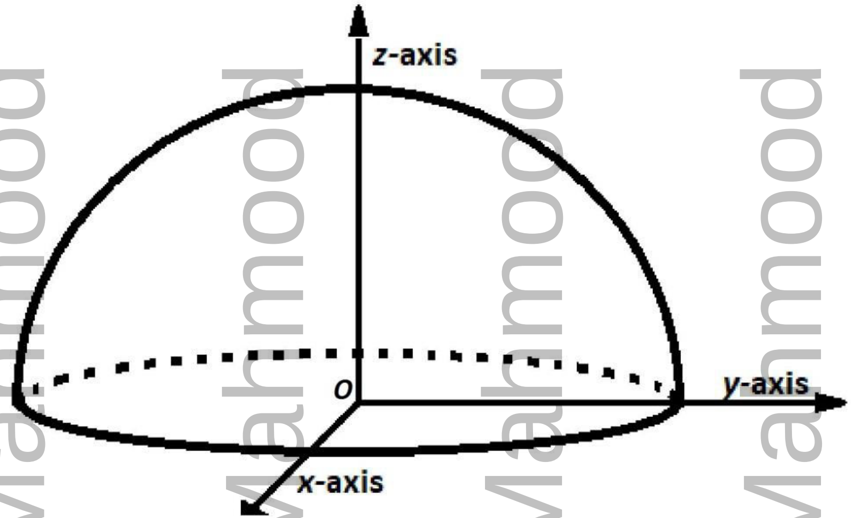
**Problem:** Show that a hoop of mass $m$ and radius $a/\sqrt{2}$ is equimomental with a circular plate of mass $m$ radius $a$.

**Proof:** The moment of inertia of a circular hoop (or ring) of mass $m$ and radius $a/\sqrt{2}$ about an axis through its centre and perpendicular to its plane is $\quad I_1 = m\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{1}{2}ma^2.$

The moment of inertia of a circular plate (or disc) of mass $m$ and radius $a$ about an axis through its centre and perpendicular to its plane is $\quad I_2 = \frac{1}{2}ma^2.$

Since, both moments of inertia are same. Therefore both systems are equimomental.

**Problem:** ***Find the (direction of) principal axes and principal moments of inertia*** of a (uniform) solid hemisphere of mass $M$ at centre of the its base.




**Solution: Moments of inertia:**

Let $M$, $a$ and $\rho$, respectively, be the mass, radius and volume mass density of the hemisphere. Choose coordinate axes as shown in figure.

Moment of inertia of typical volume element of hemisphere, with mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about $z$-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

Thus, moment of inertia of hemisphere about $z$-axis is

$$\begin{aligned} I_{zz} &= \int_{\text{Hemisphere}} (x^2 + y^2)\mathrm{d}m \\ &= \rho \int_{\text{Hemisphere}} (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \boxed{\because \quad \rho = \frac{\mathrm{d}m}{\mathrm{d}V} = \frac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} = \text{constant}} \\ &= \frac{3M}{2\pi a^3} \int_{\text{Hemisphere}} (y^2 + z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \boxed{\because \quad \rho = \frac{M}{\frac{2}{3}\pi a^3} \quad \text{(for hemisphere)}} \end{aligned}$$

To make the computation simpler, we transform the problem from Cartesian coordinates to spherical coordinates $(r,\, \theta,\, \phi)$ by using

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$x^2 + y^2 = r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) = r^2\sin^2\theta$$

$$\text{For hemisphere}: \quad 0 \le r \le a, \qquad 0 \le \theta \le \pi/2, \qquad 0 \le \phi < 2\pi$$

$$\Rightarrow \quad I_{zz} = \frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi \quad (1)$$

Where,

$$\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi/2}(3\sin\theta - \sin 3\theta) \qquad \boxed{\because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$$

$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Bigg|_{\theta=0}^{\pi/2} = \frac{1}{4}\left(3 - \frac{1}{3}\right) = \frac{2}{3} \qquad (2)$$

Using (2) in (1), we get

$$I_{zz} = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{2}{3}\right)(2\pi) = \frac{2}{5}Ma^2$$

Now,

$$I_{xx} = \int_{\text{Hemisphere}} (y^2 + z^2)\mathrm{d}m = \frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} (y^2 + z^2)\,\mathrm{d}V$$

Transforming problem in spherical coordinates $(r,\, \theta,\, \phi)$, we get

$$\begin{aligned} I_{xx} &= \frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4(\sin^3\theta\sin^2\phi + \cos^2\theta\sin\theta)\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\left(\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin^2\phi\,\mathrm{d}\phi + \int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi\right) \qquad (3) \end{aligned}$$

Where,

$$\int_{\phi=0}^{2\pi}\sin^2\phi\,\mathrm{d}\phi = \frac{1}{2}\int_{\phi=0}^{2\pi}(1-\cos 2\phi)\,\mathrm{d}\phi = \frac{1}{2}\left(\phi - \frac{1}{2}\sin 2\phi\right)\Bigg|_{\phi=0}^{2\pi} = \frac{1}{2}(2\pi) = \pi \qquad (4)$$

and

$$\int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta\,\mathrm{d}\theta = -\frac{1}{3}\cos^3\theta\Bigg|_{\theta=0}^{\pi/2} = \frac{1}{3} \qquad (5)$$

Using (2), (4) and (5), (3) gives

$$I_{xx} = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{4\pi}{3}\right) = \frac{2}{5}Ma^2$$

**Products of inertia:**

$$\begin{aligned} I_{xy} &= -\int_{\text{Hemisphere}} x\,y\,\mathrm{d}m = \frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} x\,y\,\mathrm{d}V \\ &= -\frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4\sin^2\theta\sin\phi\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= -\frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi/2}\sin^2\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi \end{aligned}$$

But

$$\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi\Bigg|_{\phi=0}^{2\pi} = 0 \qquad \Longrightarrow \qquad I_{xy} = 0$$

Now,

$$\begin{aligned} I_{xz} &= -\int_{\text{Hemisphere}} x\,z\,\mathrm{d}m = \frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} x\,y\,\mathrm{d}V \\ &= -\frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4\sin\theta\cos\theta\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= -\frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\cos\phi\,\mathrm{d}\phi \end{aligned}$$

But

$$\int_{\phi=0}^{2\pi}\cos\phi\,\mathrm{d}\phi = \sin\phi\Bigg|_{\phi=0}^{2\pi} = 0 \implies I_{xz} = 0 = I_{yz}, \qquad \boxed{\because\ I_{xz} = I_{yz}\ \text{(by symmetry)}}$$

Thus,

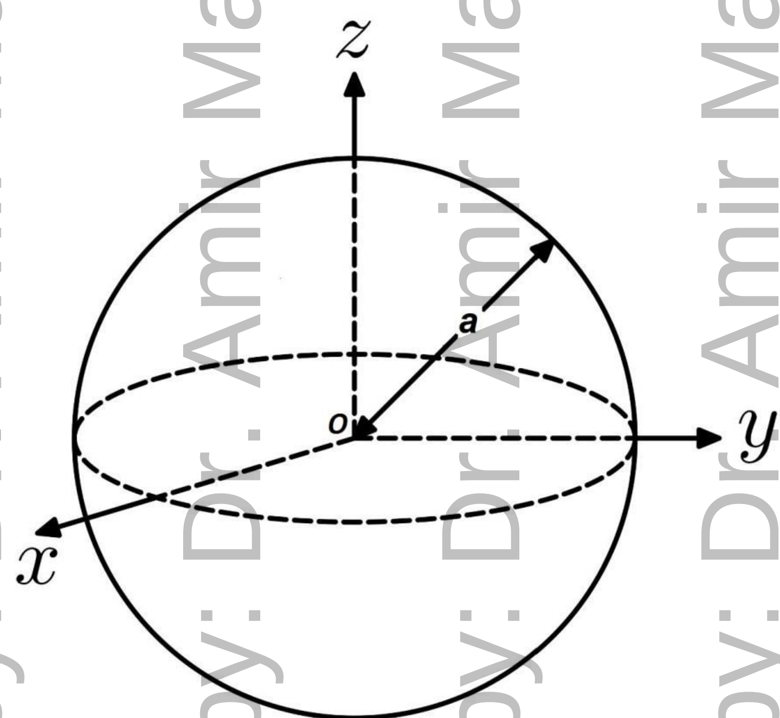
$$I_{xy} = I_{xz} = I_{yz} = 0$$

The inertia matrix with respect to coordinate system $Oxyz$ is given by

$$[I_O] = \begin{pmatrix} \frac{2}{5}Ma^2 & 0 & 0 \\ 0 & \frac{2}{5}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 \end{pmatrix}$$

Since, all products of inertia are zero, therefore coordinate axes shown in the figure are required principle axes and corresponding moments of inertia $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}Ma^2$ are principal moments of inertia.

**Problem:** ***Find the (direction of) principal axes and principal moments of inertia*** of a (uniform) solid sphere of mass $M$ at its centre.




**Solution: Moments of inertia:**
Let $M$, $a$ and $\rho$, respectively, be the mass, radius and volume mass density of the sphere. Choose coordinate axes as shown in figure.

Moment of inertia of typical volume element of sphere, with mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about $z$-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

Thus, moment of inertia of sphere about $z$-axis is

$$I_{zz} = \int_{\text{Sphere}} (x^2 + y^2)\mathrm{d}m$$

$$= \rho \int_{\text{Sphere}} (x^2 + y^2)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \qquad \boxed{\because \ \rho = \frac{\mathrm{d}m}{\mathrm{d}V} = \frac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} = \text{constant}}$$

$$= \frac{3M}{4\pi a^3} \int_{\text{Sphere}} (y^2 + z^2)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \qquad \boxed{\because \ \rho = \frac{M}{\frac{4}{3}\pi a^3} \quad \text{(for sphere)}}$$

To make the computation simpler, we transform the problem from Cartesian coordinates to spherical coordinates $(r,\, \theta,\, \phi)$ by using

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\, d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$x^2+y^2 = r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) = r^2\sin^2\theta$$

$$\text{For sphere}: \quad 0 \le r \le a, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi$$

$$\Rightarrow \quad I_{zz} = \frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi$$

Where,

$$\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi}(3\sin\theta - \sin 3\theta) \qquad \boxed{\because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$$

$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi} = \frac{1}{4}\left[\left(3-\frac{1}{3}\right) - \left(-3+\frac{1}{3}\right)\right] = \frac{4}{3}$$

Thus,

$$I_{zz} = \frac{3M}{4\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{4}{3}\right)(2\pi) = \frac{2}{5}Ma^2$$

Similarly,

$$I_{xx} = I_{yy} = \frac{2}{5}Ma^2 \qquad \boxed{\because \quad I_{xx} = I_{yy} = I_{zz} \ \text{(by symmetry)}}$$

**Products of inertia:**

$$I_{xy} = -\int_{\text{Sphere}} x\,y\,\mathrm{d}m = \frac{3M}{4\pi a^3}\int_{\text{Sphere}} x\,y\,\mathrm{d}V$$

$$= -\frac{3M}{4\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^2\theta\sin\phi\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= -\frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^2\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi$$

But

$$\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0 \qquad \Rightarrow \qquad I_{xy} = 0$$

Similarly,

$$I_{yz} = I_{xz} = 0 \qquad \boxed{\because \quad I_{xy} = I_{yz} = I_{xz} \ \text{(by symmetry)}}$$

The inertia matrix with respect to coordinate system $Oxyz$ is given by

$$[I_O] = \begin{pmatrix} \frac{2}{5}Ma^2 & 0 & 0 \\ 0 & \frac{2}{5}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 \end{pmatrix}$$

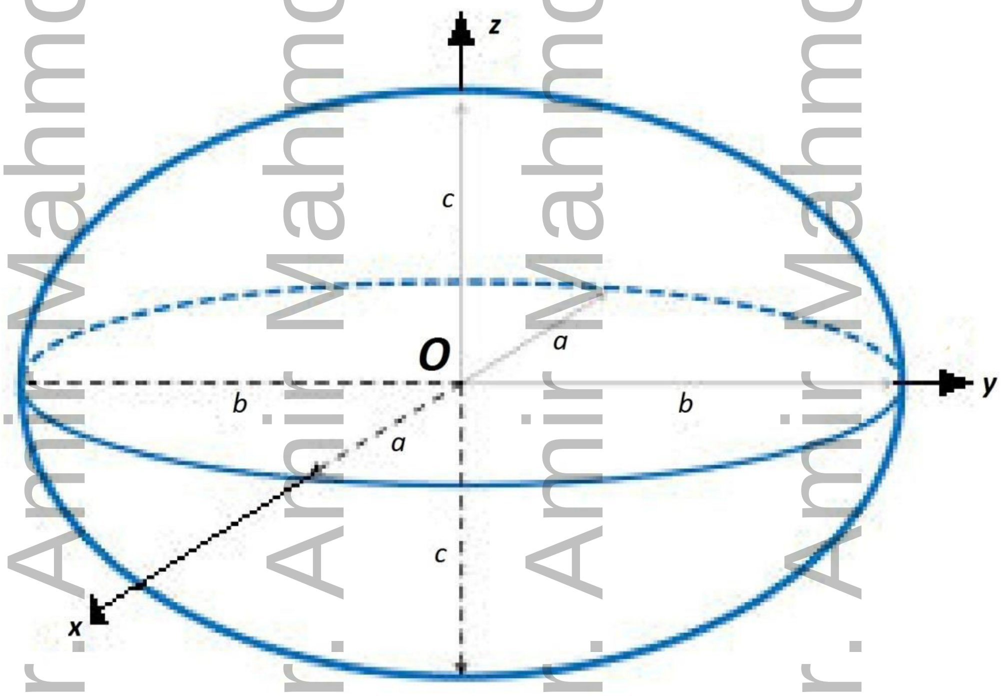
Since, all products of inertia are zero, therefore coordinate axes shown in the figure are required principle axes and corresponding moments of inertia $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}Ma^2$ are principal moments of inertia.

**Problem:** ***Find the (direction of) principal axes and principal moments of inertia*** of a (uniform) solid ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

of mass $M$ at its centre.

**Solution: Moments of inertia:**




Let $M$ and $\rho$, respectively, be the mass and volume mass density of a uniform solid ellipsoid defined by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

. Choose coordinate axes as shown in figure.

Moment of inertia of typical volume element of ellipsoid, with mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about $z$-axis is given by

$$\mathrm{d}I_{zz}=(x^2+y^2)\mathrm{d}m$$

Thus, moment of inertia of ellipsoid about $z$-axis is

$$I_{zz}=\int_{\text{Ellipsoid}}(x^2+y^2)\mathrm{d}m$$

$$=\rho\int_{\text{Ellipsoid}}(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad \boxed{\because\ \rho=\frac{\mathrm{d}m}{\mathrm{d}V}=\frac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}=\text{constant}}$$

$$=\frac{3M}{4\pi a\,b\,c}\int_{\text{Sphere}}(y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad \boxed{\because\ \rho=\frac{M}{\frac{4}{3}\pi a\,b\,c}\ \text{(for ellipsoid)}}$$

$$I_{zz}=\int_{\text{Ellipsoid}}(x^2+y^2)\mathrm{d}m \tag{6}$$

Let us substitute

$$x/a = x', \quad y/b = y', \quad z/c = z'$$

$$\Rightarrow \quad \mathrm{d}x/a = \mathrm{d}x', \quad \mathrm{d}y/b = \mathrm{d}y', \quad \mathrm{d}z/c = \mathrm{d}z', \qquad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = a\,b\,c\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

Under the above transformation, the given ellipsoid is transformed into the unit sphere

$$S : x'^2 + y'^2 + z'^2 = 1.$$

$$\Rightarrow \quad I_{zz} = \frac{3M}{4\pi a\,b\,c}\int_S (a^2 x'^2 + b^2 y'^2)(\,a\,b\,c\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z')$$
$$= \frac{3M}{4\pi}\int_S (a^2 x'^2 + b^2 y'^2)\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

$$\because \quad \int_S x'^2 \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' = \int_S y'^2 \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' \qquad \text{(by symmetry)}$$

$$\Rightarrow \quad I_{zz} = \frac{3M(a^2+b^2)}{4\pi}\int_S x'^2 \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

To make the computation simpler, we transform the problem from Cartesian coordinates ($x'$, $y'$, $z'$) to spherical coordinates ($r$, $\theta$, $\phi$) by using

$$x' = r\sin\theta\cos\phi, \qquad y' = r\sin\theta\sin\phi, \qquad z' = r\cos\theta$$

$$\mathrm{d}V = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

For unit sphere,

$$0 \le r \le 1, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi$$

$$\Rightarrow \quad I_{zz} = \frac{3M(a^2+b^2)}{4\pi}\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\cos^2\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$= \frac{3M(a^2+b^2)}{4\pi}\int_{r=0}^{1} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\cos^2\phi\,\mathrm{d}\phi$$

Where, $\displaystyle\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi}(3\sin\theta - \sin 3\theta)$ $\quad \boxed{\because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$

$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi} = \frac{1}{4}\left[\left(3-\frac{1}{3}\right) - \left(-3+\frac{1}{3}\right)\right] = \frac{4}{3}$$

and

$$\int_{\phi=0}^{2\pi}\cos^2\phi\,\mathrm{d}\phi = \frac{1}{2}\int_{\phi=0}^{2\pi}(1+\cos 2\phi)\,\mathrm{d}\phi = \frac{1}{2}\left(\phi + \frac{1}{2}\sin 2\phi\right)\Big|_{\phi=0}^{2\pi} = \frac{1}{2}(2\pi) = \pi$$

$$\Rightarrow \quad I_{zz} = \frac{3M(a^2+b^2)}{4\pi}\left(\frac{1}{5}\right)\left(\frac{4}{3}\right)(\pi) = \frac{1}{5}M(a^2+b^2)$$

Similarly,

$$I_{xx} = \frac{1}{5}M(b^2+c^2) \qquad \text{and} \qquad I_{yy} = \frac{1}{5}M(a^2+c^2)$$

**Products of inertia:**

$$\Rightarrow \quad I_{xy} = -\int_{\text{Ellipsoid}} x\,y\,\mathrm{d}m = -\,\frac{3M}{4\pi a\,b\,c}\int_{\text{Ellipsoid}} x\,y\,\mathrm{d}V$$

$$= -\,\frac{3M}{4\pi a\,b\,c}\int_{\mathrm{S}}(\,a\,b\,x'\,y')(a\,b\,c\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z')$$

$$= -\,\frac{3a\,b\,M}{4\pi}\int_{\mathrm{S}} x'\,y'\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

$$= -\,\frac{3\,a\,b\,M}{4\pi}\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^2\theta\sin\phi\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$I_{xy} = -\,\frac{3\,a\,b\,M}{4\pi}\int_{r=0}^{1} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^2\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi$$

But

$$\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0 \qquad \Longrightarrow \qquad I_{xy} = 0$$

Similarly, it is not difficult to show that

$$I_{yz} = I_{xz} = 0$$

The inertia matrix with respect to coordinate system $Oxyz$ is given by

$$[I_O] = \begin{pmatrix} \frac{1}{5}M(b^2+c^2) & 0 & 0 \\ 0 & \frac{1}{5}M(a^2+c^2) & 0 \\ 0 & 0 & \frac{1}{5}M(a^2+b^2) \end{pmatrix}$$

Since, all products of inertia are zero, therefore coordinate axes shown in the figure are required principle axes and corresponding moments of inertia $I_{xx}$, $I_{yy}$ and $I_{zz}$ are principal moments of inertia.