EXERCISE # 10.2

❖ *Find the general solution of each of the following:*

Question # 1: $(D^2+3D-4)y=15e^x$

Solution:

Given equation is

$(D^2+3D-4)y=15e^x\ ---(i)$

The characteristics equation of (i) will be

$D^2+3D-4=0$

$\Rightarrow D^2+4D-D-4=0$

$\Rightarrow D(D+4)-1(D+4)=0$

$\Rightarrow (D+4)(D-1)=0$

$\Rightarrow (D+4)=0 \text{ or } (D-1)=0$

$\Rightarrow D=-4 \text{ or } D=1$

Therefore, the complementary solution is

$y_c=c_1e^{-4x}+c_2e^x$

Now,

$$y_p=\frac{15e^x}{D^2+3D-4}$$

$$\Rightarrow y_p=\frac{15e^x}{(D+4)(D-1)}$$

$$\Rightarrow y_p=\frac{15xe^x}{(1+4)}$$

$\Rightarrow y_p=3xe^x$

Hence,

$y=y_c+y_p$

$\Rightarrow y=c_1e^{-4x}+c_2e^x+3xe^x$

is the required solution.

Question # 2: $(D^2-3D+2)y=e^x+e^{2x}$

Solution:

Given equation is

$(D^2-3D+2)y=e^x+e^{2x}\ ---(i)$

The characteristics equation of (i) will be

$D^2-3D+2=0$

$\Rightarrow D^2-2D-D+2=0$

$\Rightarrow D(D-2)-1(D-2)=0$

$\Rightarrow (D-2)(D-1)=0$

$\Rightarrow (D-2)=0 \text{ or } (D-1)=0$

$\Rightarrow D=2 \text{ or } D=1$

Therefore, the complementary solution is

$y_c=c_1e^{2x}+c_2e^x$

Now,

$$y_p=\frac{e^x+e^{2x}}{D^2-3D+2}$$

$$\Rightarrow y_p=\frac{e^x+e^{2x}}{(D-2)(D-1)}$$

$$\Rightarrow y_p=\frac{e^x}{(D-2)(D-1)}+\frac{e^{2x}}{(D-2)(D-1)}$$

$$\Rightarrow y_p=\frac{xe^x}{(1-2)}++\frac{xe^{2x}}{(2-1)}$$

$\Rightarrow y_p=-xe^x+xe^{2x}$

Hence,

$y=y_c+y_p$

$\Rightarrow y = c_1 e^{2x} + c_2 e^{x} - xe^{x} + xe^{2x}$

is the required solution.

Question # 3: $(D^2 - 2D - 3)y = 2e^x - 10\sin x$

Solution:

Given equation is

$(D^2 - 2D - 3)y = 2e^x - 10\sin x - - - (i)$

The characteristics equation of (i) *will be*

$D^2 - 2D - 3 = 0$

$\Rightarrow D^2 - 3D + D - 3 = 0$

$\Rightarrow D(D-3) + 1(D-3) = 0$

$\Rightarrow (D-3)(D+1) = 0$

$\Rightarrow (D-3) = 0 \text{ or } (D+1) = 0$

$\Rightarrow D = 3 \text{ or } D = -1$

Therefore, the complementary solution is

$y_c = c_1 e^{3x} + c_2 e^{-x}$

Now,

$$y_p = \frac{2e^x - 10\sin x}{D^2 - 2D - 3}$$

$$\Rightarrow y_p = \frac{2e^x - 10\sin x}{(D-3)(D+1)}$$

$$\Rightarrow y_p = \frac{2e^x}{(D-3)(D+1)} - \frac{10\sin x}{(D-3)(D+1)}$$

$$\Rightarrow y_p = \frac{2e^x}{(1-3)(1+1)} - \frac{10\, Im\, e^{ix}}{(D-3)(D+1)}$$

$$\Rightarrow y_p = \frac{2e^x}{(-2)(2)} - \frac{10\, Im\, e^{ix}}{(i-3)(i+1)}$$

$$\Rightarrow y_p = -\frac{e^x}{2} - \frac{10\, Im\, e^{ix}}{i^2 - 2i - 3}$$

$$\Rightarrow y_p = -\frac{e^x}{2} + \frac{5\, Im\, e^{ix}}{i+2}$$

$$\Rightarrow y_p = -\frac{e^x}{2} + \frac{5\, Im\, e^{ix}}{i+2} \times \frac{i-2}{i-2}$$

$$\Rightarrow y_p = -\frac{e^x}{2} - \frac{5\, Im(\cos x + i\sin x)(i-2)}{5}$$

$$\Rightarrow y_p = -\frac{e^x}{2} - Im(i\cos x - 2\cos x - \sin x - 2i\sin x)$$

$$\Rightarrow y_p = -\frac{e^x}{2} - \cos x + 2\sin x$$

Hence,

$y = y_c + y_p$

$$\Rightarrow y = c_1 e^{3x} + c_2 e^{-x} - \frac{e^x}{2} - \cos x + 2\sin x$$

is the required solution.

Question # 4: $(D^4 - 2D^3 + D)y = x^4 + 3x + 1$

Solution:

Given equation is

$: (D^4 - 2D^3 + D)y = x^4 + 3x + 1 - - - (i)$

The characteristics equation of (i) *will be*

$D^4 - 2D^3 + D = 0$

$\Rightarrow D(D^3 - 2D^2 + 1) = 0$

$\Rightarrow D = 0 \text{ or } (D^3 - 2D^2 + 1) = 0$

As $D = 1$ *is the root of* $D^3 - 2D^2 + 1 = 0$*. so, by using synthetic division, we have*

	1	-2	0	1
1	0	1	-1	-1
	1	-1	-1	0

The residue equation will be

$D^2 - D - 1 = 0$

$$\Rightarrow D=\frac{-(-1)\mp\sqrt{(-1)^2-4(1)(-1)}}{2(1)}$$

$$\Rightarrow D=\frac{1\mp\sqrt{5}}{2}$$

Therefore, the complementary solution is

$$y_c=c_1e^x+c_2e^{\frac{1+\sqrt{5}}{2}x}++c_3e^{\frac{1-\sqrt{5}}{2}x}$$

Now,

$$y_p=\frac{x^4+3x+1}{D^4-2D^3+D}$$

$$\Rightarrow y_p=\frac{x^4+3x+1}{D[1+(D^3-2D^2)]}$$

$$\Rightarrow y_p=\frac{1}{D}\left[1+\left(D^3-2D^2\right)\right]^{-1}\left(x^4+3x+1\right)$$

$$\Rightarrow y_p=\frac{1}{D}\left[1-\left(D^3-2D^2\right)+\left(D^3-2D^2\right)^2-\cdots\right]\left(x^4+3x+1\right)$$

$$\Rightarrow y_p=\frac{1}{D}\left[1-D^3+2D^2+4D^4+\text{neglecting higher order}\right]\left(x^4+3x+1\right)$$

$$\Rightarrow y_p=\frac{1}{D}\left[1-D^3+2D^2+4D^4\right]\left(x^4+3x+1\right)$$

$$\Rightarrow y_p=\frac{1}{D}\left[x^4+3x+1-24x+24x^2+96\right]$$

$$\Rightarrow y_p=\frac{1}{D}\left[x^4++24x^2-21x+97\right]$$

$$\Rightarrow y_p=\frac{x^5}{5}++24\frac{x^3}{3}-21\frac{x^2}{2}+97x$$

$$\Rightarrow y_p=\frac{x^5}{5}++8x^3-\frac{21}{2}x^2+97x$$

Hence,

$$y=y_c+y_p$$

$$\Rightarrow y=c_1e^x+c_2e^{\frac{1+\sqrt{5}}{2}x}++c_3e^{\frac{1-\sqrt{5}}{2}x}+\frac{x^5}{5}++8x^3-\frac{21}{2}x^2+97x$$

is the required solution.

Question # 5: $(D^3-D^2+D-1)y=4\sin x$

Solution:

Given equation is

$$\left(D^3-D^2+D-1\right)Dy=4\sin x---(i)$$

The characteristics equation of (i) *will be*

$$D^3-D^2+D-1=0$$

$$\Rightarrow D^2(D-1)+1(D-1)=0$$

$$\Rightarrow (D-1)(D^2+1)=0$$

$$\Rightarrow D-1=0 \text{ or } D^2+1=0$$

$$\Rightarrow D=1 \text{ or } D=\pm i$$

Therefore, the complementary solution is

$$y_c=c_1e^x+c_2\cos x+c_3\sin x$$

Now,

$$y_p=\frac{4\sin x}{D^3-D^2+D-1}$$

$$\Rightarrow y_p=\frac{4\,Im\,e^{ix}}{(D-1)(D^2+1)}$$

$$\Rightarrow y_p=\frac{4\,Im\,e^{ix}}{(D-1)(D+i)(D-i)}$$

$$\Rightarrow y_p=\frac{4x\,Im\,e^{ix}}{(i-1)(i+i)}$$

$$\Rightarrow y_p=\frac{2x\,Im\,e^{ix}}{i^2-i}$$

$$\Rightarrow y_p=\frac{2x\,Im\,e^{ix}}{-1-i}$$

$$\Rightarrow y_p = \frac{2x\, Im\, e^{ix}}{-1-i} \times \frac{-1+i}{-1+i}$$

$$\Rightarrow y_p = \frac{2x\, Im(\cos x + i \sin x)(-1+i)}{(-1)^2 - i^2}$$

$$\Rightarrow y_p = \frac{2x\, Im(-\cos x + i\cos x - i \sin x - \sin x)}{2}$$

$$\Rightarrow y_p = x(\cos x - \sin x)$$

Hence,

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 e^x + c_2 \cos x + c_3 \sin x + x(\cos x - \sin x)$$

$$\Rightarrow y = c_1 e^x + c_2 \cos x + c_3 \sin x + x\cos x - x \sin x$$

is the required solution.

Question # 6: $(D^3 - 2D^2 - 3D + 10)y = 40 \cos x$

Solution:

Given equation is

$$(D^3 - 2D^2 - 3D + 10)y = 40 \cos x - - - (i)$$

The characteristics equation of (i) will be

$$D^3 - 2D^2 - 3D + 10 = 0$$

As $D = -2$ is the root of $D^3 - 2D^2 - 3D + 10 = 0$. so, by using synthetic division, we have

	1	-2	-3	10
-2	0	-2	8	-10
	1	-4	5	0

The residue equation will be

$$D^2 - 4D + 5 = 0$$

$$\Rightarrow D = \frac{-(-4) \mp \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$

$$\Rightarrow D = \frac{4 \mp \sqrt{-4}}{2}$$

$$\Rightarrow D = \frac{4 \mp 2i}{2}$$

$$\Rightarrow D = 2 \mp i$$

Therefore, the complementary solution is

$$y_c = c_1 e^{-2x} + (c_2 \cos x + c_3 \sin x)e^{2x}$$

Now,

$$y_p = \frac{40 \cos x}{D^3 - 2D^2 - 3D + 10}$$

$$\Rightarrow y_p = \frac{40\, Re\, e^{ix}}{(D+2)(D^2 - 4D + 5)}$$

$$\Rightarrow y_p = \frac{40\, Re\, e^{ix}}{(i+2)(i^2 - 4i + 5)}$$

$$\Rightarrow y_p = \frac{40\, Re\, e^{ix}}{(i+2)(4-4i)}$$

$$\Rightarrow y_p = \frac{40\, Re\, e^{ix}}{4i + 4 + 8 - 8i}$$

$$\Rightarrow y_p = \frac{40\, Re\, e^{ix}}{12 - 4i}$$

$$\Rightarrow y_p = \frac{40\, Re\, e^{ix}}{4(3-i)} \times \frac{3+i}{3+i}$$

$$\Rightarrow y_p = \frac{10\, Re(\cos x + i \sin x)(3+i)}{9+1}$$

$$\Rightarrow y_p = Re(3\cos x + i \cos x + 3i \sin x - \sin x)$$

$$\Rightarrow y_p = 3\cos x - \sin x$$

Hence,

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{-2x} + (c_2 \cos x + c_3 \sin x)e^{2x} + 3\cos x - \sin x$$

is the required solution.

Question # 7: $(D^2 + 4)y = 4 \sin^2 x$

Solution:

Given equation is

$(D^2+4)y = 4\sin^2 x - - - (i)$

The characteristics equation of (i) will be

$D^2+4=0$

$\Rightarrow D^2 = -4$

$\Rightarrow D = \pm 2i$

Therefore, the complementary solution is

$y_c = c_1\cos 2x + c_2\sin 2x$

Now,

$$y_p = \frac{4\sin^2 x}{D^2+4}$$

$$\Rightarrow y_p = \frac{4\left(\frac{1-\cos 2x}{2}\right)}{D^2+4}$$

$$\Rightarrow y_p = \frac{2-2\cos 2x}{D^2+4}$$

$$\Rightarrow y_p = \frac{2}{D^2+4} - \frac{2\cos 2x}{D^2+4}$$

$$\Rightarrow y_p = \frac{2}{4\left(1+\frac{D^2}{4}\right)} - \frac{2\,Re\,e^{2ix}}{(D+2i)(D-2i)}$$

$$\Rightarrow y_p = \frac{1}{2}\left(1+\frac{D^2}{4}\right)^{-1} - \frac{2x\,Re\,e^{2ix}}{(2i+2i)}$$

$$\Rightarrow y_p = \frac{1}{2}(1) - \frac{2x\,Re(\cos 2x + i\sin 2x)}{4i} \times \frac{4i}{4i}$$

$$\Rightarrow y_p = \frac{1}{2} - \frac{2x\,Re(4i\cos 2x - 4\sin 2x)}{-16}$$

$$\Rightarrow y_p = \frac{1}{2} + \frac{2x(-4\sin 2x)}{16}$$

$$\Rightarrow y_p = \frac{1}{2} - \frac{x\sin 2x}{2}$$

$$\Rightarrow y_p = \frac{1}{2}(1 - x\sin 2x)$$

Hence,

$y = y_c + y_p$

$$\Rightarrow y = c_1\cos 2x + c_2\sin 2x + \frac{1}{2}(1 - x\sin 2x)$$

is the required solution.

Question # 8: $(D^3+D)y = 2x^2 + 4\sin x$

Solution:

Given equation is

$(D^3+D)y = 2x^2 + 4\sin x - - - (i)$

The characteristics equation of (i) will be

$D^3 + D = 0$

$\Rightarrow D(D^2+1) = 0$

$\Rightarrow D = 0, D = \pm i$

Therefore, the complementary solution is

$y_c = c_1 e^{0x} + c_2\cos x + c_3\sin x$

$\Rightarrow y_c = c_1 + c_2\cos x + c_3\sin x$

Now,

$$y_p = \frac{2x^2 + 4\sin x}{D^3+D}$$

$$\Rightarrow y_p = \frac{2x^2}{D^3+D} + \frac{4\sin x}{D^3+D}$$

$$\Rightarrow y_p = \frac{2x^2}{D(1+D^2)} + \frac{4\,Im\,e^{ix}}{D(1+D^2)}$$

$$\Rightarrow y_p = \frac{2x^2}{D}(1+D^2)^{-1} + \frac{4\,Im\,e^{ix}}{D(D+i)(D-i)}$$

$$\Rightarrow y_p = \frac{2x^2}{D}(1-D^2) + \frac{4x\,Im\,e^{ix}}{i(i+i)}$$

$$\Rightarrow y_p = \frac{1}{D}(2x^2-4) + \frac{4x\,Im(\cos x + i\sin x)}{-2}$$

$$\Rightarrow y_p = \frac{2}{3}x^3 - 4x - 2x\sin x$$

Hence,

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$y = y_c + y_p$

$$\Rightarrow y = c_1 + c_2 \cos x + c_3 \sin x + \frac{2}{3}x^3 - 4x - 2x \sin x$$

is the required solution.

Question # 9: $(D^4 + D^2)y = 3x^2 + 6\sin x - 2\cos x$

Solution:

Given equation is

$$(D^4 + D^2)y = 3x^2 + 6\sin x - 2\cos x --- (i)$$

The characteristics equation of (i) will be

$$D^4 + D^2 = 0$$

$$\Rightarrow D^2(D^2 + 1) = 0$$

$$\Rightarrow D = 0, 0, D = \pm i$$

Therefore, the complementary solution is

$$y_c = c_1 e^{0x} + c_2 x e^{0x} + c_3 \cos x + c_4 \sin x$$

$$\Rightarrow y_c = c_1 + c_2 x + c_3 \cos x + c_4 \sin x$$

Now,

$$y_p = \frac{3x^2 + 6\sin x - 2\cos x}{D^4 + D^2}$$

$$\Rightarrow y_p = \frac{3x^2}{D^4 + D^2} + \frac{6\sin x}{D^4 + D^2} - \frac{2\cos x}{D^4 + D^2}$$

$$\Rightarrow y_p = \frac{3x^2}{D^2(1 + D^2)} + \frac{6\, Im\, e^{ix}}{D^2(D^2 + 1)} - \frac{2\, Re\, e^{ix}}{D^2(D^2 + 1)}$$

$$\Rightarrow y_p = \frac{3x^2}{D^2}(1 + D^2)^{-1} + \frac{6\, Im\, e^{ix}}{D^2(D + i)(D - i)} - \frac{2\, Re\, e^{ix}}{D^2(D + i)(D - i)}$$

$$\Rightarrow y_p = \frac{3x^2}{D^2}(1 - D^2) + \frac{6x\, Im\, e^{ix}}{i^2(i + i)} - \frac{2x\, Re\, e^{ix}}{i^2(i + i)}$$

$$\Rightarrow y_p = \frac{1}{D^2}(3x^2 - 6) - \frac{3x Im(\cos x + i\sin x)}{i} + \frac{x\, Re(\cos x + i\sin x)}{i}$$

$$\Rightarrow y_p = \frac{x^4}{4} - 3x^2 - \frac{3x Im(i\cos x - \sin x)}{i^2} + \frac{x\, Re(i\cos x - \sin x)}{i^2}$$

$$\Rightarrow y_p = \frac{x^4}{4} - 3x^2 + 3x Im(i\cos x - \sin x) - x\, Re(i\cos x - \sin x)$$

$$\Rightarrow y_p = \frac{x^4}{4} - 3x^2 + 3x\cos x + x\sin x$$

Hence,

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 + c_2 x + c_3 \cos x + c_4 \sin x + \frac{x^4}{4} - 3x^2 + 3x\cos x + x\sin x$$

is the required solution.

Question # 10: $(D^2 - 2D + 4)y = e^x \cos x$

Solution:

Given equation is

$$(D^2 - 2D + 4)y = e^x \cos x --- (i)$$

The characteristics equation of (i) will be

$$D^2 - 2D + 4 = 0$$

$$\Rightarrow D = \frac{-(-2) \pm \sqrt{4 - 16}}{2}$$

$$\Rightarrow D = \frac{2 \pm \sqrt{-12}}{2}$$

$$\Rightarrow D = \frac{2 \pm 2\sqrt{3}i}{2}$$

$$\Rightarrow D = 1 \pm \sqrt{3}i$$

Therefore, the complementary solution is

$$y_c = \left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right)e^x$$

Now,

$$y_p = \frac{e^x \cos x}{D^2 - 2D + 4}$$

$$\Rightarrow y_p = \frac{e^x \cos x}{(D+1)^2 - 2(D+1) + 4} \quad (By\ exponential\ shift)$$

$$\Rightarrow y_p = \frac{e^x \cos x}{D^2 + 2D + 1 - 2D - 2 + 4}$$

$$\Rightarrow y_p = \frac{e^x \cos x}{D^2 + 3}$$

$$\Rightarrow y_p = \frac{e^x\, Re\, e^{ix}}{D^2 + 3}$$

$$\Rightarrow y_p = \frac{e^x\, Re(\cos x + i \sin x)}{-1 + 3}$$

$$\Rightarrow y_p = \frac{e^x \cos x}{2}$$

Hence,

$$y = y_c + y_p$$

$$\Rightarrow y = \left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right)e^x + \frac{e^x \cos x}{2}$$

is the required solution.

Question # 11: $\left(D^3 - D^2 + 3D + 5\right)y = e^x \sin 2x$

Solution:

Given equation is

$$\left(D^3 - D^2 + 3D + 5\right)y = e^x \sin 2x \text{ --- } (i)$$

The characteristics equation of (i) will be

$$D^3 - D^2 + 3D + 5 = 0$$

$D = -1$ is a root of characteristics equation. So we use synthetic division in order to find the other roots of synthetic division.

	1	-1	3	5
-1	0	-1	2	-5
	1	-2	5	0

Now, the residual equation will be

$$D^2 - 2D + 5 = 0$$

$$\Rightarrow D = \frac{-(-2) \pm \sqrt{4 - 20}}{2}$$

$$\Rightarrow D = \frac{2 \pm \sqrt{-16}}{2}$$

$$\Rightarrow D = \frac{2 \pm 4i}{2}$$

$$\Rightarrow D = 1 \pm 2i$$

Therefore, the complementary solution is

$$y_c = c_1 e^{-x} + (c_2 \cos 2x + c_3 \sin 2x)e^x$$

Now,

$$y_p = \frac{e^x \sin 2x}{D^3 - D^2 + 3D + 5}$$

$$\Rightarrow y_p = \frac{e^x \sin 2x}{(D+1)^3 - (D+1)^2 + 3(D+1) + 5}$$

$$(By\ exponential\ shift)$$

$$\Rightarrow y_p = \frac{e^x \sin 2x}{D^3 + 1 + 3D^2 + 3D - D^2 - 2D - 1 + 3D + 3 + 5}$$

$$\Rightarrow y_p = \frac{e^x \sin 2x}{D^3 + 2D^2 + 4D + 8}$$

$$\Rightarrow y_p = \frac{e^x \sin 2x}{D^2(D+2) + 4(D+2)}$$

$$\Rightarrow y_p = \frac{e^x \sin 2x}{(D+2)(D^2+4)}$$

$$\Rightarrow y_p = \frac{e^x\, Im\, e^{2ix}}{(D+2)(D+2i)(D-2i)}$$

$$\Rightarrow y_p = \frac{xe^x \, Im\, e^{2ix}}{(2i+2)(2i+2i)}$$

$$\Rightarrow y_p = \frac{xe^x \, Im\, e^{2ix}}{8i(i+1)}$$

$$\Rightarrow y_p = \frac{xe^x \, Im\, e^{2ix}}{8(i-1)}$$

$$\Rightarrow y_p = \frac{xe^x \, Im\, e^{2ix}}{8(i-1)} \times \frac{i+1}{i+1}$$

$$\Rightarrow y_p = \frac{xe^x \, Im(\cos 2x + i \sin 2x)\,(i+1)}{8(i^2-1)}$$

$$\Rightarrow y_p = \frac{xe^x \, Im(i\cos 2x + \cos 2x - \sin 2x + i \sin 2x)}{8(-2)}$$

$$\Rightarrow y_p = -\frac{xe^x(\cos 2x + \sin 2x)}{16}$$

Hence,

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{-x} + (c_2 \cos 2x + c_3 \sin 2x)e^x - \frac{xe^x(\cos 2x + \sin 2x)}{16}$$

is the required solution.

Question # 12: $\left(D^3 - 7D - 6\right)y = e^{2x}(1+x)$

Solution:

Given equation is

$$\left(D^3 - 7D - 6\right)y = e^{2x}(1+x) \; --- \; (i)$$

The characteristics equation of (i) ***will be***

$$D^3 - 7D - 6 = 0$$

$D = -1$ ***is a root of characteristics equation. So we use synthetic division in order to find the other roots of characteristics equation.***

	1	0	-7	-6
-1	0	-1	1	6
	1	-1	-6	0

Now, the residual equation will be

$$D^2 - D + 6 = 0$$

$$\Rightarrow D^2 - 3D + 2D - 6 = 0$$

$$\Rightarrow D(D-3) + 2(D-3) = 0$$

$$\Rightarrow (D-3)(D+2) = 0$$

$$\Rightarrow D = 3 \text{ or } D = -2$$

Therefore, the complementary solution is

$$y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$$

Now,

$$y_p = \frac{e^{2x}(1+x)}{D^3 - 7D - 6}$$

$$\Rightarrow y_p = \frac{e^{2x}(1+x)}{(D+2)^3 - 7(D+2) - 6}$$

$$(By\ exponential\ shift)$$

$$\Rightarrow y_p = \frac{e^{2x}(1+x)}{D^3 + 8 + 6D^2 + 12D - 7D - 14 - 6}$$

$$\Rightarrow y_p = \frac{e^{2x}(1+x)}{D^3 + 6D^2 + 5D - 12}$$

$$\Rightarrow y_p = -\frac{e^{2x}(1+x)}{12\left(1 - \dfrac{D^3 + 6D^2 + 5D}{12}\right)}$$

$$\Rightarrow y_p = -\frac{e^{2x}(1+x)}{12}\left(1 - \frac{D^3 + 6D^2 + 5D}{12}\right)^{-1}$$

$$\Rightarrow y_p = -\frac{e^{2x}(1+x)}{12}\left(1 + \frac{D^3 + 6D^2 + 5D}{12} + \cdots\right)$$

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$$\Rightarrow y_p = -\frac{e^{2x}(1+x)}{12}\left(1+\frac{5D}{12} + \text{neglecting higher power factors}\right)$$

$$\Rightarrow y_p = -\frac{e^{2x}(1+x)}{12}\left(1+\frac{5D}{12}\right)$$

$$\Rightarrow y_p = -\frac{e^{2x}}{12}\left(1+x+\frac{5}{12}\right)$$

$$\Rightarrow y_p = -\frac{e^{2x}}{12}\left(x+\frac{17}{12}\right)$$

Hence,

$$y = y_c + y_p$$

$$\Rightarrow y = c_1e^{-x} + c_2e^{-2x} + c_3e^{3x} - \frac{e^{2x}}{12}\left(x+\frac{17}{12}\right)$$

is the required solution.

Question # 13: $(D^2-7D+12)y = e^{2x}(x^3-5x^2)$

Solution:

Given equation is

$$(D^2-7D+12)y = e^{2x}(x^3-5x^2) - - - (i)$$

The characteristics equation of (i) *will be*

$$D^2-7D+12=0$$

$$\Rightarrow D^2-3D-4D+12=0$$

$$\Rightarrow D(D-3)-4(D-3)=0$$

$$\Rightarrow (D-3)(D-4)=0$$

$$\Rightarrow D=3 \text{ or } D=4$$

Therefore, the complementary solution is

$$y_c = c_1e^{3x} + c_2e^{4x}$$

Now,

$$y_p = \frac{e^{2x}(x^3-5x^2)}{D^2-7D+12}$$

$$\Rightarrow y_p = \frac{e^{2x}(x^3-5x^2)}{(D+2)^2-7(D+2)+12}$$

$$(By\ exponential\ shift)$$

$$\Rightarrow y_p = \frac{e^{2x}(x^3-5x^2)}{D^2+4+4D-7D-14+12}$$

$$\Rightarrow y_p = \frac{e^{2x}(x^3-5x^2)}{D^2-3D+2}$$

$$\Rightarrow y_p = \frac{e^{2x}(x^3-5x^2)}{2\left(1+\frac{D^2-3D}{2}\right)}$$

$$\Rightarrow y_p = \frac{e^{2x}}{2}\left(1+\frac{D^2-3D}{2}\right)^{-1}(x^3-5x^2)$$

$$\Rightarrow y_p = \frac{e^{2x}}{2}\left(1-\frac{D^2-3D}{2}+\frac{(D^2-3D)^2}{4}-\frac{(D^2-3D)^3}{8}+\cdots\right)(x^3-5x^2)$$

$$\Rightarrow y_p = \frac{e^{2x}}{2}\left(1-\frac{D^2}{2}+\frac{3D}{2}+\frac{9D^2}{4}-\frac{6D^3}{4}+\frac{27D^3}{8}+\cdots\right)(x^3-5x^2)\quad \text{neglecting greater powers}$$

$$\Rightarrow y_p = \frac{e^{2x}}{2}\left(1+\frac{15D^3}{8}+\frac{7D^2}{4}+\frac{3D}{2}\right)(x^3-5x^2)$$

$$\Rightarrow y_p = \frac{e^{2x}}{2}\left(x^3-5x^2+\frac{90}{8}+\frac{42}{4}x-\frac{70}{4}+\frac{9x^2}{2}-\frac{30x}{2}\right)$$

$$\Rightarrow y_p = \frac{e^{2x}}{2}\left(x^3-\frac{x^2}{2}-\frac{18x}{4}-\frac{50}{8}\right)$$

$$\Rightarrow y_p = \frac{e^{2x}}{2}\left(x^3-\frac{x^2}{2}-\frac{9x}{2}-\frac{25}{4}\right)$$

$$\Rightarrow y_p = \frac{e^{2x}}{8}(4x^3-2x^2-18x-25)$$

Hence,

$$y = y_c + y_p$$

$$\Rightarrow y = c_1e^{3x} + c_2e^{4x} + \frac{e^{2x}}{8}(4x^3 - 2x^2 - 18x - 25)$$

is the required solution.

Question # 14: $(D^4 + 8D^2 - 9)y = 9x^3 + 5\cos 2x$

Solution:

Given equation is

$$(D^4 + 8D^2 - 9)y = 9x^3 + 5\cos 2x - - - (i)$$

The characteristics equation of (i) *will be*

$$D^4 + 8D^2 - 9 = 0$$

$$\Rightarrow D^4 + 9D^2 - D^2 - 9 = 0$$

$$\Rightarrow D^2(D^2 + 9) - 1(D^2 + 9) = 0$$

$$\Rightarrow (D^2 + 9)(D^2 - 1) = 0$$

$$\Rightarrow D = \pm 3i \text{ or } D = \pm 1$$

Therefore, the complementary solution is

$$y_c = c_1e^{x} + c_2e^{-x} + c_3\cos 3x + c_4\sin 3x$$

Now,

$$y_p = \frac{9x^3 + 5\cos 2x}{D^4 + 8D^2 - 9}$$

$$\Rightarrow y_p = \frac{9x^3}{D^4 + 8D^2 - 9} + \frac{5\cos 2x}{D^4 + 8D^2 - 9}$$

$$\Rightarrow y_p = \frac{9x^3}{-9\left(1 - \frac{D^4 + 8D^2}{9}\right)} + \frac{5\,\mathrm{Re}\,e^{2ix}}{(D^2+9)(D^2-1)}$$

$$\Rightarrow y_p = -x^3\left(1 - \frac{D^4 + 8D^2}{9}\right)^{-1} + \frac{5\,\mathrm{Re}\,e^{2ix}}{((2i)^2 + 9)((2i)^2 - 1)}$$

$$\Rightarrow y_p = -x^3\left(1 + \frac{D^4 + 8D^2}{9} + \cdots\right) + \frac{5\,\mathrm{Re}\,e^{2ix}}{(-4+9)(-4-1)}$$

$$\Rightarrow y_p = -x^3\left(1 + \frac{8D^2}{9} + \cdots \textit{neglecting terms}\right) + \frac{5\,\mathrm{Re}(\cos 2x + i\sin 2x)}{(5)(-5)}$$

$$\Rightarrow y_p = -x^3 - \frac{8}{9}(6x) + \frac{\cos 2x}{(-5)}$$

$$\Rightarrow y_p = -x^3 - \frac{16}{3}x - \frac{\cos 2x}{5}$$

Hence,

$$y = y_c + y_p$$

$$\Rightarrow y = c_1e^{x} + c_2e^{-x} + c_3\cos 3x + c_4\sin 3x - x^3 - \frac{16}{3}x - \frac{\cos 2x}{5}$$

is the required solution. **Question # 15:**

$$(D^4 + 3D^2 - 4)y = \sinh x - \cos^2 x$$

Solution:

Given equation is

$$(D^4 + 3D^2 - 4)y = \sinh x - \cos^2 x - - - (i)$$

The characteristics equation of (i) *will be*

$$D^4 + 3D^2 - 4 = 0$$

$D = 1$ & $D = -1$ *are the roots of characteristics equation. So we use synthetic division in order to find the other roots of characteristics equation.*

	1	0	3	0	-4
1	0	1	1	4	4
	1	1	4	4	0
-1	0	-1	0	-4	
	1	0	4	0	

Now, the residual equation will be

$$D^2 + 4 = 0$$

$$\Rightarrow D^2 = -4$$

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$\Rightarrow D = \pm 2i$

Therefore, the complementary solution is

$y_c = c_1 e^x + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x$

Now,

$$y_p = \frac{\sinh x - \cos^2 x}{D^4 + 3D^2 - 4}$$

$$\Rightarrow y_p = \frac{\sinh x}{D^4 + 3D^2 - 4} - \frac{\cos^2 x}{D^4 + 3D^2 - 4}$$

$$\Rightarrow y_p = \frac{\frac{e^x - e^{-x}}{2}}{(D+1)(D-1)(D+2i)(D-2i)} - \frac{\frac{1+\cos 2x}{2}}{D^4 + 3D^2 - 4}$$

$$\Rightarrow y_p = y_{p_1} - y_{p_2} - - - (a)$$

Consider,

$$y_{p_1} = \frac{\frac{e^x - e^{-x}}{2}}{(D+1)(D-1)(D+2i)(D-2i)}$$

$$\Rightarrow y_{p_1} = \frac{1}{2}\left[\frac{e^x}{(D+1)(D-1)(D+2i)(D-2i)} - \frac{e^{-x}}{(D+1)(D-1)(D+2i)(D-2i)}\right]$$

$$\Rightarrow y_{p_1} = \frac{1}{2}\left[\frac{e^x}{(D+1)(D-1)(D+2i)(D-2i)} - \frac{e^{-x}}{(D+1)(D-1)(D+2i)(D-2i)}\right]$$

$$\Rightarrow y_{p_1} = \frac{1}{2}\left[\frac{xe^x}{(1+1)(1+2i)(1-2i)} - \frac{xe^{-x}}{(-1-1)(-1+2i)(-1-2i)}\right]$$

$$\Rightarrow y_{p_1} = \frac{1}{2}\left[\frac{xe^x}{10} - \frac{xe^{-x}}{10}\right]$$

$$\Rightarrow y_{p_1} = \frac{x}{2}\left[\frac{e^x}{10} - \frac{e^{-x}}{10}\right]$$

$$\Rightarrow y_{p_1} = \frac{x}{2}\left[\frac{e^x - e^{-x}}{10}\right]$$

$$\Rightarrow y_{p_1} = \frac{x}{10}\left[\frac{e^x - e^{-x}}{2}\right]$$

$$\Rightarrow y_{p_1} = \frac{x \sinh x}{10}$$

Now consider,

$$y_{p_2} = \frac{\frac{1+\cos 2x}{2}}{D^4 + 3D^2 - 4}$$

$$y_{p_2} = \frac{1}{2}\left[\frac{1}{D^4 + 3D^2 - 4} + \frac{\cos 2x}{D^4 + 3D^2 - 4}\right]$$

$$\Rightarrow y_{p_2} = \frac{1}{2}\left[\frac{1}{-4\left(1 - \frac{D^4 + 3D^2}{4}\right)} + \frac{\operatorname{Re} e^{2ix}}{(D+1)(D-1)(D+2i)(D-2i)}\right]$$

$$\Rightarrow y_{p_2} = \frac{1}{2}\left[-\frac{1}{4}\left(1 - \frac{D^4 + 3D^2}{4}\right)^{-1} + \frac{x \operatorname{Re} e^{2ix}}{(2i+1)(2i-1)(2i+2i)}\right]$$

$$\Rightarrow y_{p_2} = \frac{1}{2}\left[-\frac{1}{4}(1) + \frac{x \operatorname{Re} e^{2ix}}{(-5)(4i)}\right]$$

$$\Rightarrow y_{p_2} = \frac{1}{2}\left[-\frac{1}{4} - \frac{x \operatorname{Re} e^{2ix}}{20i}\right]$$

$$\Rightarrow y_{p_2} = \frac{1}{2}\left[-\frac{1}{4} + \frac{x \operatorname{Re}(\cos 2x + i \sin 2x) i}{20}\right]$$

$$\Rightarrow y_{p_2} = \frac{1}{2}\left[-\frac{1}{4} + \frac{x \operatorname{Re}(i \cos 2x - \sin 2x)}{20}\right]$$

$$\Rightarrow y_{p_2} = \frac{1}{2}\left[-\frac{1}{4} + \frac{x(-\sin 2x)}{20}\right]$$

$$\Rightarrow y_{p_2} = \frac{1}{2}\left[-\frac{1}{4} - \frac{x \sin 2x}{20}\right]$$

$$\Rightarrow y_{p_2} = -\frac{1}{8} + \frac{x \sin 2x}{40}$$

Thus equation (a) becomes

$$\Rightarrow y_p = \frac{x\sinh x}{10} - \frac{1}{8} + \frac{x\sin 2x}{40}$$

Hence,

$$y = y_c + y_p$$

$$\Rightarrow y = c_1e^x + c_2e^{-x} + c_3\cos 2x + c_4\sin 2x + \frac{x\sinh x}{10} - \frac{1}{8} + \frac{x\sin 2x}{40}$$

is the required solution.

❖ ***Solve the initial value problem.***

Question # 16:

$$y'' - 8y' + 15y = 9xe^{2x} \qquad y(0) = 5, y'(0) = 10$$

Solution:

Given equation is

$$y'' - 8y' + 15y = 9xe^{2x} - - - (i)$$

The characteristics equation of (i) will be

$$D^2 - 8D + 15 = 0$$

$$\Rightarrow D^2 - 3D - 5D + 15 = 0$$

$$\Rightarrow D(D-3) - 5(D-3) = 0$$

$$\Rightarrow (D-3)(D-5) = 0$$

$$\Rightarrow D = 3 \text{ or } D = 5$$

Therefore, the complementary solution is

$$y_c = c_1e^{3x} + c_2e^{5x}$$

Now,

$$y_p = \frac{9xe^{2x}}{D^2 - 8D + 15}$$

$$\Rightarrow y_p = \frac{9xe^{2x}}{(D+2)^2 - 8(D+2) + 15}$$

$$(By\ exponential\ shift)$$

$$\Rightarrow y_p = \frac{9xe^{2x}}{D^2 + 4 + 4D - 8D - 16 + 15}$$

$$\Rightarrow y_p = \frac{9xe^{2x}}{D^2 - 4D + 3}$$

$$\Rightarrow y_p = \frac{9xe^{2x}}{3\left(1 + \frac{D^2 - 4D}{3}\right)}$$

$$\Rightarrow y_p = 3e^{2x}\left(1 + \frac{D^2 - 4D}{3}\right)^{-1} x$$

$$\Rightarrow y_p = 3e^{2x}\left(1 - \frac{D^2 - 4D}{3} + \left(\frac{D^2 - 4D}{3}\right)^2 - \cdots\right) x$$

$$\Rightarrow y_p = 3e^{2x}\left(1 + \frac{4D}{3} + neglecting\ higher\ powers\right) x$$

$$\Rightarrow y_p = 3e^{2x}\left(1 + \frac{4D}{3}\right) x$$

$$\Rightarrow y_p = 3e^{2x}\left(x + \frac{4}{3}\right)$$

$$\Rightarrow y_p = 3xe^{2x} + 4e^{2x}$$

Hence,

$$y = y_c + y_p$$

$$\Rightarrow y = c_1e^{3x} + c_2e^{5x} + 3xe^{2x} + 4e^{2x} - - - (ii)$$

Applying $y(0) = 5$ on (ii), we have

$$5 = c_1 + c_2 + 4$$

$$\Rightarrow c_1 + c_2 = 1$$

$$\Rightarrow c_2 = 1 - c_1 - - - (a)$$

Differentiating (ii) $w.r.t$ "x", we have

$$y' = 3c_1e^{3x} + 5c_2e^{5x} + 3e^{2x} + 6xe^{2x} + 8e^{2x}$$

$$\Rightarrow y' = 3c_1e^{3x} + 5c_2e^{5x} + 6xe^{2x} + 11e^{2x} - - - (iii)$$

Applying $y'(0) = 10$ on (iii), we have

$$10 = 3c_1 + 5c_2 + 11$$

$$\Rightarrow 3c_1 + 5c_2 = -1$$

$$\Rightarrow 3c_1 + 5(1 - c_1) = -1$$

$$\Rightarrow 3c_1 + 5 - 5c_1 = -1$$

$$\Rightarrow -2c_1 = -6$$

$$\Rightarrow c_1 = 3$$

Now $(a) \Rightarrow$

$$c_2 = 1 - 3$$

$$\Rightarrow c_2 = -2$$

Hence,

$$y = 3e^{3x} - 2e^{5x} + 3xe^{2x} + 4e^{2x}$$

is the required solution.

Question # 17:

$$y'' - 4y' + 13y = 8\sin 3x \qquad y(0) = 1, y'(0) = 2$$

Solution:

Given equation is

$$y'' - 4y' + 13y = 8\sin 3x - - - (i)$$

The characteristics equation of (i) will be

$$D^2 - 4D + 13 = 0$$

$$\Rightarrow D = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$\Rightarrow D = \frac{4 \pm \sqrt{-36}}{2}$$

$$\Rightarrow D = \frac{4 \pm 6i}{2}$$

$$\Rightarrow D = 2 \pm 3i$$

Therefore, the complementary solution is

$$y_c = (c_1 \cos 3x + c_2 \sin 3x)e^{2x}$$

Now,

$$y_p = \frac{8\sin 3x}{D^2 - 4D + 13}$$

$$\Rightarrow y_p = \frac{8\,\mathrm{Im}\,e^{3ix}}{D^2 - 4D + 13}$$

$$\Rightarrow y_p = \frac{8\,\mathrm{Im}\,e^{3ix}}{-9 - 12i + 13}$$

$$\Rightarrow y_p = \frac{8\,\mathrm{Im}\,e^{3ix}}{4 - 12i}$$

$$\Rightarrow y_p = \frac{8\,\mathrm{Im}\,e^{3ix}}{4(1 - 3i)} \times \frac{1 + 3i}{1 + 3i}$$

$$\Rightarrow y_p = \frac{2\,\mathrm{Im}(\cos 3x + i\sin 3x)(1 + 3i)}{1 + 9}$$

$$\Rightarrow y_p = \frac{2\,\mathrm{Im}(\cos 3x + 3i\cos 3x + i\sin 3x - 3\sin 3x)}{10}$$

$$\Rightarrow y_p = \frac{3\cos 3x + \sin 3x}{5}$$

Hence,

$$y = y_c + y_p$$

$$\Rightarrow y = (c_1 \cos 3x + c_2 \sin 3x)e^{2x} + \frac{3\cos 3x + \sin 3x}{5} - - - (ii)$$

Applying $y(0) = 1$ on (ii), we have

$$1 = c_1 + \frac{3}{5}$$

$$\Rightarrow c_1 = 1 - \frac{3}{5}$$

$$\Rightarrow c_1 = \frac{2}{5} - - - (a)$$

Differentiating (ii) $w.r.t$ "x", we have

$$y' = 2(c_1 \cos 3x + c_2 \sin 3x)e^{2x} + (-3c_1 \sin 3x + 3c_2 \cos 3x)e^{2x} + \frac{-9\sin 3x + 3\cos 3x}{5} - -(iii)$$

Applying $y'(0) = 2$ on (iii), we have

$$2 = 2c_1 + 3c_2 + \frac{3}{5}$$

$$\Rightarrow 2 = \frac{4}{5} + 3c_2 + \frac{3}{5}$$

$$\Rightarrow 2 = \frac{7}{5} + 3c_2$$

$$\Rightarrow 2 - \frac{7}{5} = 3c_2$$

$$\Rightarrow \frac{3}{5} = 3c_2$$

$$\Rightarrow c_2 = \frac{1}{5}$$

Hence,

$$y = \left(\frac{2}{5}\cos 3x + \frac{1}{5}\sin 3x\right)e^{2x} + \frac{3\cos 3x + \sin 3x}{5}$$

$$\Rightarrow y = \frac{1}{5}[(\sin 3x + 2\cos 3x)e^{2x} + 3\cos 3x + \sin 3x]$$

is the required solution.

Question # 18: $y'' - 4y = 2 - 8x \qquad y(0) = 0, y'(0) = 5$

Solution:

Given equation is

$$y'' - 4y = 2 - 8x - - - (i)$$

The characteristics equation of (i) will be

$$D^2 - 4 = 0$$

$$\Rightarrow D^2 = 4$$

$$\Rightarrow D = \pm 2$$

Therefore, the complementary solution is

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

Now,

$$y_p = \frac{2 - 8x}{D^2 - 4}$$

$$\Rightarrow y_p = -\frac{2 - 8x}{4\left(1 - \frac{D^2}{4}\right)}$$

$$\Rightarrow y_p = -\frac{2 - 8x}{4}\left(1 - \frac{D^2}{4}\right)^{-1}$$

$$\Rightarrow y_p = -\frac{2 - 8x}{4}\left(1 + \frac{D^2}{4} + \cdots\right)$$

$$\Rightarrow y_p = -\frac{2 - 8x}{4}(1)$$

$$\Rightarrow y_p = -\frac{2 - 8x}{4}$$

Hence,

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} - \frac{2 - 8x}{4} - - - (ii)$$

Applying $y(0) = 0$ on (ii), we have

$$0 = c_1 + c_2 - \frac{2}{4}$$

$$\Rightarrow c_1 + c_2 = \frac{1}{2}$$

$$\Rightarrow c_2 = \frac{1}{2} - c_1 - - - (a)$$

Differentiating (ii) $w.r.t$ "x", we have

$$y' = 2c_1 e^{2x} - 2c_2 e^{-2x} - \frac{1}{4}(-8)$$

$$y' = 2c_1 e^{2x} - 2c_2 e^{-2x} + 2 - - - (iii)$$

Applying $y'(0) = 5$ on (iii), we have

$$5 = 2c_1 - 2c_2 + 2$$

$$\Rightarrow 2c_1 - 2c_2 = 3$$

$$\Rightarrow 2c_1 - 2\left(\frac{1}{2} - c_1\right) = 3$$

$$\Rightarrow 4c_1 = 3 + 1$$

$$\Rightarrow c_1 = 1$$

Now $(a) \Rightarrow$

$$c_2 = \frac{1}{2} - (1)$$

$$\Rightarrow c_2 = -\frac{1}{2}$$

Hence,

$$y = e^{2x} - \frac{1}{2}e^{-2x} - \frac{2-8x}{4}$$

$$\Rightarrow y = e^{2x} - \frac{1}{2}e^{-2x} + 2x - \frac{1}{2}$$

is the required solution.

Question # 19: $y'' + y = x\sin x$

$$y(0) = 1, y'(0) = 2$$

Solution:

Given equation is

$$y'' + y = x\sin x - - - (i)$$

The characteristics equation of (i) *will be*

$$D^2 + 1 = 0$$

$$\Rightarrow D^2 = -1$$

$$\Rightarrow D = \pm i$$

Therefore, the complementary solution is

$$y_c = c_1\cos x + c_2\sin x$$

Now,

$$y_p = \frac{x\sin x}{D^2+1}$$

$$\Rightarrow y_p = \frac{\operatorname{Im} xe^{ix}}{D^2+1}$$

$$\Rightarrow y_p = \operatorname{Im}\frac{xe^{ix}}{(D+i)^2+1} \quad (by\ exponential\ shift)$$

$$\Rightarrow y_p = \operatorname{Im}\frac{xe^{ix}}{D^2-1+2Di+1}$$

$$\Rightarrow y_p = \operatorname{Im}\frac{xe^{ix}}{D^2+2Di}$$

$$\Rightarrow y_p = \operatorname{Im}\frac{xe^{ix}}{2iD\left(1+\frac{D}{2i}\right)}$$

$$\Rightarrow y_p = \operatorname{Im}\frac{e^{ix}}{2iD}\left(1+\frac{D}{2i}\right)^{-1}x$$

$$\Rightarrow y_p = \operatorname{Im}\frac{e^{ix}}{2iD}\left(1-\frac{D}{2i}\right)x$$

$$\Rightarrow y_p = \operatorname{Im}e^{ix}\frac{1}{D}\frac{1}{2i}\left(x-\frac{1}{2i}\right)$$

$$\Rightarrow y_p = \operatorname{Im}e^{ix}\frac{1}{D}\left(\frac{x}{2i}-\frac{1}{4i^2}\right)$$

$$\Rightarrow y_p = \operatorname{Im}e^{ix}\frac{1}{D}\left(\frac{x}{2i}+\frac{1}{4}\right)$$

$$\Rightarrow y_p = \operatorname{Im}e^{ix}\left(\frac{x^2}{4i}+\frac{x}{4}\right)$$

$$\Rightarrow y_p = \operatorname{Im}(\cos x + i\sin x)\left(\frac{x^2}{4i}+\frac{x}{4}\right)$$

$$\Rightarrow y_p = \operatorname{Im}\left(\frac{x^2}{4i}\cos x + \frac{x}{4}\cos x + \frac{x^2}{4i}i\sin x + \frac{x}{4}i\sin x\right)$$

$$\Rightarrow y_p = \operatorname{Im}\left(-\frac{ix^2}{4}\cos x + \frac{x}{4}\cos x + \frac{x^2}{4}\sin x + \frac{x}{4}i\sin x\right)$$

$$\Rightarrow y_p = -\frac{x^2}{4}\cos x + \frac{x}{4}\sin x$$

$$\Rightarrow y_p = \frac{x\sin x}{4} - \frac{x^2\cos x}{4}$$

Hence,

$$y = y_c + y_p$$

$$y = c_1\cos x + c_2\sin x + \frac{x\sin x}{4} - \frac{x^2\cos x}{4} - -(ii)$$

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Applying $y(0) = 1$ on (ii), we have

$$\mathbf{1 = c_1}$$

Differentiating (ii) $w.r.t$ "x", we have

$$y' = -c_1 \sin x + c_2 \cos x + \frac{\sin x}{4} + \frac{x \cos x}{4} - \frac{2x \cos x}{4} + \frac{x^2 \sin x}{4} - - - (iii)$$

Applying $y'(0) = 2$ on (iii), we have

$$\mathbf{2 = c_2}$$

Hence,

$$y = \cos x + 2 \sin x + \frac{x \sin x}{4} - \frac{x^2 \cos x}{4}$$

Is the required solution.

Question # 20:

$$y''' + 3y'' + 7y' + 5y = 16e^{-x} \cos 2x$$

$$y(0) = 2, y'(0) = -4, y''(0) = -2$$

Solution:

Given equation is

$$y''' + 3y'' + 7y' + 5y = 16e^{-x} \cos 2x - - - (i)$$

The characteristics equation of (i) is

$$D^3 + 3D^2 + 7D + 5 = 0$$

$D = 1$ & $D = -1$ are the roots of characteristics equation. So we use synthetic division in order to find the other roots of characteristics equation.

	1	3	7	5
-1	0	-1	-2	-5
	1	2	5	0

Now, the residual equation will be

$$D^2 + 2D + 5 = 0$$

$$\Rightarrow D = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)}$$

$$\Rightarrow D = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$\Rightarrow D = \frac{-2 \pm \sqrt{-16}}{2}$$

$$\Rightarrow D = \frac{-2 \pm 4i}{2}$$

$$\Rightarrow D = -1 \pm 2i$$

Therefore, the complementary solution is

$$y_c = (c_1 \cos 2x + c_2 \sin 2x)e^{-x} + c_3 e^{-x}$$

Now,

$$y_p = \frac{16e^{-x} \cos 2x}{D^3 + 3D^2 + 7D + 5}$$

First, we will use the exponential shift and then use the process which is used in above questions and finally we will reach

$$y_p = -2e^{-x} x \cos 2x$$

Hence,

$$y = y_c + y_p$$

$$y = (c_1 \cos 2x + c_2 \sin 2x)e^{-x} + c_3 e^{-x} - 2e^{-x} x \cos 2x - -(ii)$$

Since initial boundary value conditions are given. We will use that conditions and obtain the final result as below

$$y = 2e^{-x} \cos 2x - 2e^{-x} x \cos 2x$$

Is the required solution.