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Exercise No. 9.5

In problems 1–10 find eqs. for the tangent plane & the normal line to the given surface at the indicated pt. P:

Q1 $4x^2 - y^2 + 3z^2 = 10$; $P(2,-3,1)$

Sol. Given eq. is

$$4x^2 - y^2 + 3z^2 - 10 = 0$$

Let $f(x,y,z) = 4x^2 - y^2 + 3z^2 - 10$

then $\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= 8x\hat{i} - 2y\hat{j} + 6z\hat{k}$$

So $\text{grad} f = 16\hat{i} + 6\hat{j} + 6\hat{k}$ at pt. $P(2,-3,1)$

Now eq. of tangent plane to surface is

$$16(x-2) + 6(y+3) + 6(z-1) = 0$$

or $16x - 32 + 6y + 18 + 6z - 6 = 0$

$$16x + 6y + 6z - 20 = 0$$

or $\boxed{8x + 3y + 3z - 10 = 0}$

& eq. of normal line to surface is

$$\frac{x-2}{16} = \frac{y+3}{6} = \frac{z-1}{6}$$

or $\frac{x-2}{8} = \frac{y+3}{3} = \frac{z-1}{3}$

Q2 $x^2 + y^2 + z^2 = 14$; $P(1,-2,3)$

Sol. Given eq. is

$$x^2 + y^2 + z^2 - 14 = 0$$

Here $f(x,y,z) = x^2 + y^2 + z^2 - 14$

$\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$\text{grad } f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$= 2\hat{i} - 4\hat{j} + 6\hat{k} \quad \text{at } P(1,-2,3)$$

Now eq. of tangent plane to surface is

$$2(x-1) - 4(y+2) + 6(z-3) = 0$$

$$2x - 2 - 4y - 8 + 6z - 18 = 0$$

$$2x - 4y + 6z - 28 = 0$$

or $$\boxed{x - 2y + 3z - 14 = 0}$$

& eq. of normal line to surface is

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{3}$$

Q3 $\quad 9x^2 + 4y^2 - z^2 = 36 \qquad P(2,3,6)$

Sol. Given surface is

$$9x^2 + 4y^2 - z^2 - 36 = 0$$

Here $f(x,y,z) = 9x^2 + 4y^2 - z^2 - 36$

then $$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 18x\hat{i} + 8y\hat{j} - 2z\hat{k}$$

$$\text{grad } f = 36\hat{i} + 24\hat{j} - 12\hat{k} \quad \text{at pt } P(2,3,6)$$

Now eq. of tangent plane to given surface is

$$36(x-2) + 24(y-3) - 12(z-6) = 0$$

$$3(x-2) + 2(y-3) - (z-6) = 0$$

$$3x - 6 + 2y - 6 - z + 6 = 0$$

$$\boxed{3x + 2y - z - 6 = 0}$$

& eq. of normal line to surface is

$$\frac{x-2}{36} = \frac{y-3}{24} = \frac{z-6}{-12}$$

or $$\boxed{\frac{x-2}{3} = \frac{y-3}{2} = \frac{z-6}{-1}}$$

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Q4 $\quad x^2-2y^2-z^2=4 \quad , P(-6,2,\sqrt{24})$

Sol. Given surface is

$$x^2-2y^2-z^2-4=0$$

Here $f(x,y,z) = x^2-2y^2-z^2-4$

Then $\text{grad} f = \frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}$

$$= 2x\hat{i}-4y\hat{j}-2z\hat{k}$$

or $\text{grad} f_0 = -12\hat{i}-8\hat{j}-2\sqrt{24}\,\hat{k}$ at pt. $P(-6,2,\sqrt{24})$

Now eq. of tangent plane to surface is

$$-12(x+6)-8(y-2)-2\sqrt{24}\,(z-\sqrt{24})=0$$

$$-12(x+6)-8(y-2)-2(2\sqrt{6})(z-\sqrt{24})=0$$

Dividing both sides by -4

$$3(x+6)+2(y-2)+\sqrt{6}\,(z-\sqrt{24})=0$$

$$3x+18+2y-4+\sqrt{6}z-\sqrt{24\times 6}=0$$

$$3x+2y+\sqrt{6}z+14-\sqrt{144}=0$$

$$3x+2y+\sqrt{6}z+14-12=0$$

$$\boxed{3x+2y+\sqrt{6}\,z+2=0}$$

Now eq. of normal to surface is

$$\frac{x+6}{3}=\frac{y-2}{2}=\frac{z-\sqrt{24}}{\sqrt{6}}$$

Q5 $\quad z=x^2+y^2 \quad ; P(-2,1,5)$

Sol. Given surface is

$z=x^2+y^2$

or $x^2+y^2-z=0$

Here $f(x,y,z) = x^2+y^2-z$

then $\text{grad} f = \frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}$

$= 2x\hat{i} + 2y\hat{j} - \hat{k}$

or $\text{grad} f = -4\hat{i} + 2\hat{j} - \hat{k}$ at pt. $P(-2, 1, 5)$

Now eq. of tangent plane to given surface is

$-4(x+2) + 2(y-1) - 1(z-5) = 0$

$-4x - 8 + 2y - 2 - z + 5 = 0$

$-4x + 2y - z - 5 = 0$

$\boxed{4x - 2y + z + 5 = 0}$

& eq. of normal line to surface is

$$\frac{x+2}{4} = \frac{y-1}{-2} = \frac{z-5}{1}$$

Q6 $xz = 4$; $P(-2, 2, -2)$

Sol: Given surface is

$xz = 4$

or $xz - 4 = 0$,

Here $f(x, y, z) = xz - 4$

Then $\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$= z\hat{i} + 0\hat{j} + x\hat{k}$

or $\text{grad} f = -2\hat{i} + 0\hat{j} - 2\hat{k}$ at pt. $P(-2, 2, -2)$

Now eq. of tangent plane to surface is

$-2(x+2) + 0(y-2) - 2(z+2) = 0$

$x + 2 + z + 2 = 0$

or $\boxed{x + z + 4 = 0}$

& eq. of normal line to surface is

$$\frac{x+2}{1} = \frac{y-2}{0} = \frac{z+3}{1}$$

Q7 $x^2+z^2 = \frac{a^2}{h^2}y^2$; $P\left(\frac{a}{\sqrt{2}}, h, \frac{a}{\sqrt{2}}\right)$

Soln. Given surface is

$$x^2+z^2-\frac{a^2}{h^2}y^2 = 0$$

Here $f(x,y,z) = x^2-\frac{a^2}{h^2}y^2+z^2$

Then

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}$$

$$= 2x\hat{i}-\frac{2a^2y}{h^2}\hat{j}+2z\hat{k}$$

$$= \frac{2a}{\sqrt{2}}\hat{i}-\frac{2a^2}{h^2}\cdot h\hat{j}+\frac{2a}{\sqrt{2}}\hat{k}$$

$$\text{grad } f_{P_0} = \sqrt{2}a\,\hat{i}-\frac{2a^2}{h}\hat{j}+\sqrt{2}a\,\hat{k} \quad \text{at } P\left(\frac{a}{\sqrt{2}}, h, \frac{a}{\sqrt{2}}\right)$$

Now eq. of tangent plane to surface is

$$\sqrt{2}a\left(x-\frac{a}{\sqrt{2}}\right)-\frac{2a^2}{h}(y-h)+\sqrt{2}a\left(z-\frac{a}{\sqrt{2}}\right)=0$$

Dividing both sides by $\sqrt{2}a$

$$x-\frac{a}{\sqrt{2}}-\frac{\sqrt{2}a}{h}(y-h)+z-\frac{a}{\sqrt{2}}=0$$

$$x-\frac{a}{\sqrt{2}}-\frac{\sqrt{2}a}{h}y+\sqrt{2}a+z-\frac{a}{\sqrt{2}}=0$$

$$x-\frac{\sqrt{2}a}{h}y+z-\frac{2a}{\sqrt{2}}+\sqrt{2}a=0$$

$$x-\frac{\sqrt{2}a}{h}y+z-\cancel{\sqrt{2}a}+\cancel{\sqrt{2}a}=0$$

or $\boxed{x-\frac{\sqrt{2}a}{h}y+z=0}$

& eq. of normal to surface is

$$\frac{x-\frac{a}{\sqrt{2}}}{1} = \frac{y-h}{-\frac{\sqrt{2}a}{h}} = \frac{z-\frac{a}{\sqrt{2}}}{1}$$

Q.8 $z = e^x \cos y$; $P(0, \frac{\pi}{2}, 0)$

Sol. Given surface is

$$e^x \cos y - z = 0$$

Here $f(x,y,z) = e^x \cos y - z$

Then

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= e^x \cos y\,\hat{i} - e^x \sin y\,\hat{j} - \hat{k}$$

$$= e^0 \cos \pi/2\,\hat{i} - e^0 \sin \pi/2\,\hat{j} - \hat{k}$$

$$\text{grad } f = 0\hat{i} - \hat{j} - \hat{k} \quad \text{at } P(0, \tfrac{\pi}{2}, 0)$$

Now eq. of tangent plane to surface is

$$0(x-0) - 1(y - \pi/2) - 1(z-0) = 0$$

$$-y + \pi/2 - z = 0$$

$$\boxed{y + z - \pi/2 = 0}$$

& eq. of normal line to surface is

$$\frac{x-0}{0} = \frac{y - \pi/2}{1} = \frac{z-0}{1}$$

or $$\frac{x}{0} = \frac{y - \pi/2}{1} = \frac{z}{1}$$

Q9 $x = \ln\left(\frac{y}{2z}\right)$; $P(0,2,1)$

Sol. Given surface is

$$x = \ln\left(\frac{y}{2z}\right)$$

or $$x - \ln\left(\frac{y}{2z}\right) = 0$$

Here $f(x,y,z) = x - \ln\left(\frac{y}{2z}\right) = x - \ln y + \ln 2 + \ln z$

then $$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \hat{i} - \frac{1}{y}\hat{j} + \frac{1}{z}\hat{k}$$

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$\text{grad} f = \hat{i} - \frac{1}{2}\hat{j} + \hat{k}$ at pt: $P(0,2,1)$

Now eq. of tangent plane to surface is

$$1(x-0) - \frac{1}{2}(y-2) + 1(z-1) = 0$$

$$x - \frac{1}{2}(y-2) + z - 1 = 0$$

$$2x - (y-2) + 2(z-1) = 0$$

$$2x - y + \cancel{2} + 2z - \cancel{2} = 0$$

$$\boxed{2x - y + 2z = 0}$$

& eq. of normal to surface is

$$\frac{x}{2} = \frac{y-2}{-1} = \frac{z-1}{2}$$

Q10 $x^{2/3} + y^{2/3} + z^{4/3} = 9$; $P(1,8,-8)$

Sol. Given surface is

$$x^{2/3} + y^{2/3} + z^{2/3} - 9 = 0$$

Here $f(x,y,z) = x^{2/3} + y^{2/3} + z^{2/3} - 9$

then $\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= \frac{2}{3}x^{-1/3}\hat{i} + \frac{2}{3}y^{-1/3}\hat{j} + \frac{2}{3}z^{-1/3}\hat{k}$$

$$= \frac{2}{3x^{1/3}}\hat{i} + \frac{2}{3y^{1/3}}\hat{j} + \frac{2}{3z^{1/3}}\hat{k}$$

$$= \frac{2}{3(1)}\hat{i} + \frac{2}{3(2)}\hat{j} + \frac{2}{3(-2)}\hat{k}$$

$\text{grad} f = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{1}{3}\hat{k}$ at $P(1,8,-8)$

Now eq. of tangent plane to surface is

$$\frac{2}{3}(x-1) + \frac{1}{3}(y-8) - \frac{1}{3}(z+8) = 0$$

$$2(x-1) + y - 8 - (z+8) = 0$$

$$2x - 2 + y - 8 - z - 8 = 0$$

$$\boxed{2x + y - z - 18 = 0}$$

& eq. of normal to surface is

$$\frac{x-1}{2} = \frac{y-8}{1} = \frac{z+8}{-1}$$

<u>Q11</u> Find the pt. on $x^2-2y^2-4z^2=16$ at which the tangent plane is parallel to the plane $4x-2y+4z=5$.

<u>Sol.</u> Given surface is

$$x^2-2y^2-4z^2=16$$

Let $P(x_1,y_1,z_1)$ be the req. pt. on the given surface then it will satisfy the eq.

$$x_1^2-2y_1^2-4z_1^2=16 \quad \text{——— (1)}$$

Here $f(x,y,z) = x^2-2y^2-4z^2-16$

then $\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= 2x\hat{i} - 4y\hat{j} - 8z\hat{k}$$

$\text{grad} f = 2x_1\hat{i} - 4y_1\hat{j} - 8z_1\hat{k}$ at pt. $P(x_1,y_1,z_1)$

Now eq. of tangent plane to surface at pt. P is

$$2x_1(x-x_1) - 4y_1(y-y_1) - 8z_1(z-z_1) = 0$$

$$x_1(x-x_1) - 2y_1(y-y_1) - 4z_1(z-z_1) = 0$$

$$x_1x - 2y_1y - 4z_1z - x_1^2 + 2y_1^2 + 4z_1^2 = 0$$

$$x_1x - 2y_1y - 4z_1z - (x_1^2 - 2y_1^2 - 4z_1^2) = 0$$

$$\boxed{x_1x - 2y_1y - 4z_1z - 16 = 0} \quad \text{using (1)}$$

Since this plane is parallel to $4x-2y+4z=5$

So $\frac{x_1}{4} = \frac{-2y_1}{-2} = \frac{-4z_1}{4} = t$

$\Rightarrow x_1 = 4t, \; y_1 = t, \; z_1 = -t$

Since $P(x_1, y_1, z_1)$ lies on given surface so

$$x_1^2 - 2y_1^2 - 4z_1^2 = 16$$

$$16t^2 - 2t^2 - 4t^2 = 16$$

$$10t^2 = 16$$

$$t^2 = \frac{16}{10}$$

$$\boxed{t = \pm \frac{4}{\sqrt{10}}}$$

So req. pt. is

$$P(x_1, y_1, z_1) = P(4t, t, -t)$$

$$= P\left(\pm \frac{16}{\sqrt{10}}, \pm \frac{4}{\sqrt{10}}, \mp \frac{4}{\sqrt{10}}\right)$$

or $P\left(\frac{16}{\sqrt{10}}, \frac{4}{\sqrt{10}}, -\frac{4}{\sqrt{10}}\right)$ & $P'\left(-\frac{16}{\sqrt{10}}, -\frac{4}{\sqrt{10}}, \frac{4}{\sqrt{10}}\right)$

<u>Q12</u> Two surfaces are said to be tangent at a common pt. P if each has the same tangent plane at P. Show that the surfaces $x^2 + z^2 + 4y = 0$ & $x^2 + y^2 + z^2 - 6z + 7 = 0$ are tangent at $P(0, -1, 2)$.

<u>Sol.</u> Given surfaces are

$$x^2 + z^2 + 4y = 0 \quad \text{——— (1)}$$

$$x^2 + y^2 + z^2 - 6z + 7 = 0 \quad \text{——— (2)}$$

Here $f(x, y, z) = x^2 + z^2 + 4y$

then $\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= 2x\hat{i} + 4\hat{j} + 2z\hat{k}$$

$$\text{grad} f = 0\hat{i} + 4\hat{j} + 4\hat{k} \quad \text{at pt. } P(0, -1, 2)$$

Eq. of tangent plane to surface f is

$0(x-0)+1(y+1)+4(z-2)=0$

$0+y+1+z-2=0$

$\boxed{y+z-1=0}$

Now

$g(x,y,z) = x^2+y^2+z^2-6z+7$

then grad g $= \frac{\partial g}{\partial x}\hat{i}+\frac{\partial g}{\partial y}\hat{j}+\frac{\partial g}{\partial z}\hat{k}$

$= 2x\hat{i}+2y\hat{j}+(2z-6)\hat{k}$

grad g $= 0\hat{i}-2\hat{j}-2\hat{k}$ at pt. $P(0,-1,2)$

Now eq. of tangent plane to surface g is

$0(x-0)-2(y+1)-2(z-2)=0$

$0+y+1+z-2=0$

$\boxed{y+z-1=0}$

Hence both surfaces have a common tangent plane at pt. P. So given surfaces are tangent at pt. P.

Q13 Show that the sphere $x^2+y^2+z^2=18$ & the Cone $x^2+z^2=(y-6)^2$ are tangent along their intersection.

Sol. Given surfaces are

$x^2+y^2+z^2=18$ ——— (1)

& $x^2+z^2-(y-6)^2=0$ ——— (2)

First we find their pt. of intersection

Subt. (2) from (1)

$y^2+(y-6)^2=18$

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$$y^2 + y^2 - 12y + 36 - 18 = 0$$
$$2y^2 - 12y + 18 = 0$$
$$y^2 - 6y + 9 = 0$$
$$(y-3)^2 = 0$$
$$\Rightarrow \boxed{y = 3}$$

Let $P(x_1, y_1, z_1)$ be the pt. of intersection of given surfaces then $\boxed{y_1 = 3}$

Now $f(x,y,z) = x^2 + y^2 + z^2 - 18$

then $\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$\text{grad} f = 2x_1\hat{i} + 2y_1\hat{j} + 2z_1\hat{k}$ at pt. $P(x_1, y_1, z_1)$

$\text{grad} f = 2x_1\hat{i} + 6\hat{j} + 2z_1\hat{k}$ is a normal vector to surface f at pt. P.

Now $g(x,y,z) = x^2 - (y-6)^2 + z^2$

then $\text{grad} g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$

$$= 2x\hat{i} - 2(y-6)\hat{j} + 2z\hat{k}$$

$= 2x_1\hat{i} - 2(y_1 - 6)\hat{j} + 2z_1\hat{k}$ at pt. $P(x_1, y_1, z_1)$

$= 2x_1\hat{i} - 2(3-6)\hat{j} + 2z_1\hat{k}$ as $y_1 = 3$

$\text{grad} g = 2x_1\hat{i} + 6\hat{j} + 2z_1\hat{k}$

is a normal vector to surface g at pt. P

Since both surfaces $f(x,y,z)$ & $g(x,y,z)$ have same normal vectors at pt. P.

So the surfaces are tangent along their intersection

Q Show that the surfaces $z = 16 - x^2 - y^2$ & $63z = x^2 + y^2$ intersect orthogonally.

Sol. Given surfaces are

$$x^2 + y^2 + z - 16 = 0 \text{ ——— } (1)$$

$$\& \quad x^2 + y^2 - 63z = 0 \text{ ——— } (2)$$

To find pt. of intersection of (1) & (2)

subt. (2) from (1)

$$z + 63z - 16 = 0$$

$$64z = 16$$

$$\boxed{z = \tfrac{1}{4}}$$

Put in (2)

$$x^2 + y^2 - 63(1/4) = 0$$

$$\text{or} \quad \boxed{x^2 + y^2 = \frac{63}{4}}$$

let $P(x_1, y_1, z_1)$ be the pt. of intersection of surfaces

where $z_1 = 1/4$ & $x_1^2 + y_1^2 = \frac{63}{4}$

Now $f(x, y, z) = x^2 + y^2 + z - 16$

then $\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= 2x\hat{i} + 2y\hat{j} + \hat{k}$$

$\Rightarrow \text{grad} f = 2x_1\hat{i} + 2y_1\hat{j} + \hat{k}$ at pt. $P(x_1, y_1, z_1)$

Now $g(x, y, z) = x^2 + y^2 - 63z$

then $\text{grad}\, g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$

$$= 2x\hat{i} + 2y\hat{j} - 63\hat{k}$$

$\Rightarrow \text{grad}\, g = 2x_1\hat{i} + 2y_1\hat{j} - 63\hat{k}$ at pt. $P(x_1, y_1, z_1)$

Now $f_x \cdot g_x + f_y \cdot g_y + f_z \cdot g_z$

$= (2x_1)(2x_1) + (2y_1)(2y_1) + (1)(-63)$

$= 4x_1^2 + 4y_1^2 - 63$

$= 4(x_1^2 + y_1^2) - 63$

$= 4\left(\frac{63}{4}\right) - 63$

$= 63 - 63$

$= 0$

Hence the given surfaces intersect orthogonally.

Q15. Prove that all normal lines of the sphere $x^2+y^2+z^2 = a^2$ pass through the centre of sphere.

Sol: Given eq. of sphere is

$$x^2+y^2+z^2 = a^2$$

Let $P(x_1, y_1, z_1)$ be any pt. on the sphere

Now $f(x,y,z) = x^2+y^2+z^2 - a^2$

Then $\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\Rightarrow \text{grad} f = 2x_1\hat{i} + 2y_1\hat{j} + 2z_1\hat{k}$ at $P(x_1, y_1, z_1)$

Now eq. of normal to sphere at pt. P is

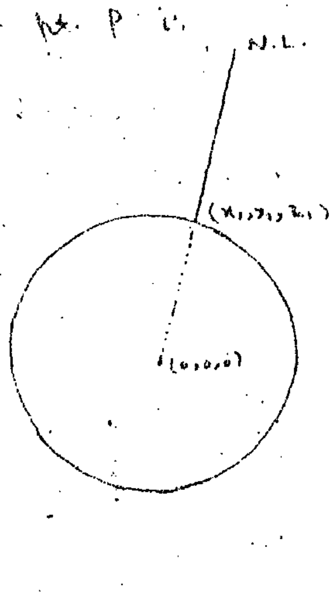
$$\frac{x-x_1}{2x_1} = \frac{y-y_1}{2y_1} = \frac{z-z_1}{2z_1}$$

or $\frac{x-x_1}{x_1} = \frac{y-y_1}{y_1} = \frac{z-z_1}{z_1}$

Put $(x,y,z) = (0,0,0)$

$$\frac{0-x_1}{x_1} = \frac{0-y_1}{y_1} = \frac{0-z_1}{z_1}$$

$$-1 = -1 = -1$$



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Since eq. of normal to sphere is satisfied by centre of sphere. So, the normal line passes through centre of sphere.

As $P(x_1, y_1, z_1)$ is any pt. on the sphere, so all normal lines to sphere pass through centre of sphere.

Q16 Show that the ellipsoid $\frac{y^2}{16} + \frac{x^2}{12} + \frac{z^2}{12} = 1$ & the hyperboloid $\frac{y^2}{3} - x^2 - z^2 = 1$ intersect orthogonally.

Sol: Given surfaces are

$$\frac{y^2}{16} + \frac{1}{12}(x^2+z^2) = 1 \qquad \text{--------- (1)}$$

$$\& \quad \frac{y^2}{3} - (x^2+z^2) = 1 \qquad \text{--------- (2)}$$

To find pt of intersection, solving (1) & (2)

from (2) $\quad x^2+z^2 = \frac{y^2}{3} - 1$

Put in (1)

$$\frac{y^2}{16} + \frac{1}{12}\left(\frac{y^2}{3} - 1\right) = 1$$

$$\frac{y^2}{16} + \frac{y^2}{36} - \frac{1}{12} = 1$$

$$\frac{y^2}{16} + \frac{y^2}{36} = 1 + \frac{1}{12}$$

$$\frac{y^2}{16} + \frac{y^2}{36} = \frac{13}{12}$$

Multiplying both sides by 144

$$9y^2 + 4y^2 = 13 \times 12$$

$$13y^2 = 13 \times 12$$

$$y^2 = 12$$

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or $\boxed{y = \pm\sqrt{12}}$ 64

let $P(x_1, y_1, z_1)$ be the pt. of intersection where $y_1 = \pm\sqrt{12}$

Now $f(x, y, z) = \frac{y^2}{16} + \frac{1}{12}(x^2 + z^2) - 1$

Now $\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$= \frac{x}{6}\hat{i} + \frac{y}{8}\hat{j} + \frac{z}{6}\hat{k}$

$= \frac{x_1}{6}\hat{i} + \frac{y_1}{8}\hat{j} + \frac{z_1}{6}\hat{k}$ at $P(x_1, y_1, z_1)$

Now $g(x, y, z) = \frac{y^2}{3} - x^2 - z^2 - 1$

then $\text{grad}\, g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$

$= -2x\hat{i} + \frac{2y}{3}\hat{j} - 2z\hat{k}$

$= -2x_1\hat{i} + \frac{2}{3}y_1\hat{j} - 2z_1\hat{k}$ at $P(x_1, y_1, z_1)$

Now

$f_x \cdot g_x + f_y \cdot g_y + f_z \cdot g_z$

$= \left(\frac{x_1}{6}\right)(-2x_1) + \left(\frac{y_1}{8}\right)\left(\frac{2}{3}y_1\right) + \left(\frac{z_1}{6}\right)(-2z_1)$

$= -\frac{x_1^2}{3} + \frac{y_1^2}{12} - \frac{z_1^2}{3}$

$= \frac{y_1^2}{12} - \frac{1}{3}(x_1^2 + z_1^2)$

$= \frac{y_1^2}{12} - \frac{1}{3}\left(\frac{y_1^2}{3} - 1\right)$

$= \frac{12}{12} - \frac{1}{3}\left(\frac{12}{3} - 1\right)$

$= 1 - \frac{1}{3}(4 - 1)$

$= 1 - \frac{1}{3}(3) = 1 - 1 = 0$

Hence given surfaces intersect orthogonally.

Q17 Find the pt. on $z = 4x^2 + 9y^2$ at which the normal line is parallel to the line through $A(-2,4,3)$ & $B(5,-1,2)$. 65

Sol:- Given surface is

$$z = 4x^2 + 9y^2$$

let $P(x_1, y_1, z_1)$ be the req. pt. on the surface.

So $z_1 = 4x_1^2 + 9y_1^2$ ——— (1)

Now $f(x,y,z) = 4x^2 + 9y^2 - z$

then $\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= 8x\hat{i} + 18y\hat{j} - \hat{k}$$

or $\text{grad} f = 8x_1\hat{i} + 18y_1\hat{j} - \hat{k}$ at $P(x_1, y_1, z_1)$

Hence d.rs. of the normal to given surface are

$8x_1, 18y_1, -1$

Now d.rs. of line AB are $5+2, -1-4, 2-3$

or $7, -5, -1$

Since the normal line is parallel to line AB

So their drs. are proportional

$$\Rightarrow \frac{8x_1}{7} = \frac{18y_1}{-5} = \frac{-1}{-1}$$

$$\Rightarrow \frac{8x_1}{7} = -\frac{18y_1}{5} = 1$$

$$\Rightarrow x_1 = \frac{7}{8}, \quad y_1 = -\frac{5}{18}$$

As $z_1 = 4x_1^2 + 9y_1^2$ from eq. (1)

$$= 4 \cdot \frac{49}{64} + 9 \cdot \frac{25}{324}$$

$$= \frac{49}{16} + \frac{25}{36}$$

$$= \frac{441 + 100}{144} = \frac{541}{144}$$

So required pt. is $P\left(\frac{7}{8}, -\frac{5}{18}, \frac{541}{144}\right)$

Q18 Where and at what angle do the Cone $x^2+y^2 = \frac{1}{2}z^2$ & the Cylinder $x^2+y^2=4$ intersect ?

Sol. Given surfaces are

$$x^2+y^2 = \frac{1}{2}z^2 \quad \text{---------- (1)}$$
$$x^2+y^2 = 4 \quad \text{---------- (2)}$$

To find pt. of intersection, solving (1) & (2)

from (2) $x^2+y^2=4$

Put in (1)

$$4 = \frac{1}{2}z^2$$
$$z^2 = 8$$
$$\boxed{z = \pm 2\sqrt{2}}$$

Hence the two surfaces intersect in the planes.

$z = 2\sqrt{2}$ & $z = -2\sqrt{2}$

Let $P(x_1, y_1, z_1)$ be the pt. of intersection of (1) & (2)

Then $z_1 = \pm 2\sqrt{2}$

Now $f(x,y,z) = x^2+y^2-\frac{1}{2}z^2$

then $\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$= 2x\hat{i} + 2y\hat{j} - z\hat{k}$

$\Rightarrow \text{grad} f = 2x_1\hat{i} + 2y_1\hat{j} - z_1\hat{k}$ at $P(x_1,y_1,z_1)$

Now $g(x,y,z) = x^2+y^2-4$

then $\text{grad}\, g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$

$= 2x\hat{i} + 2y\hat{j} + 0\hat{k}$

$= 2x_1\hat{i} + 2y_1\hat{j} + 0\hat{k}$ at $P(x_1,y_1,z_1)$

Let θ be the angle b/w. the surfaces then 61

$$\cos\theta = \frac{f_x\cdot g_x + f_y\cdot g_y + f_z\cdot g_z}{\sqrt{f_x^2+f_y^2+f_z^2}\ \sqrt{g_x^2+g_y^2+g_z^2}}$$

$$= \frac{(2x_1)(2x_1)+(2y_1)(2y_1)+(-z_1)(0)}{\sqrt{4x_1^2+4y_1^2+z_1^2}\ \sqrt{4x_1^2+4y_1^2+0^2}}$$

$$= \frac{4x_1^2+4y_1^2}{\sqrt{4(x_1^2+y_1^2)+z_1^2}\ \sqrt{4(x_1^2+y_1^2)}}$$

$$= \frac{4(x_1^2+y_1^2)}{\sqrt{4(x_1^2+y_1^2)+z_1^2}\ \sqrt{4(x_1^2+y_1^2)}}$$

$$= \frac{4(4)}{\sqrt{4(4)+8}\ \sqrt{4(4)}}$$

$\because$ (x_1, y_1, z_1) lies on $x^2+y^2=4$

So $x_1^2+y_1^2=4$

$$= \frac{16}{\sqrt{24}\sqrt{16}}$$

$$= \frac{16}{2\sqrt{6}\cdot 4}$$

$$= \frac{2}{\sqrt{6}}$$

$$\frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{3}}$$

$$= \frac{\sqrt{2}}{\sqrt{3}}$$

$$\cos\theta = \sqrt{\frac{2}{3}}$$

$\Rightarrow \theta = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$ is the req. angle.

Q19 For the surface defined by parametric eqs. 68

$x = 2\cosh u \cos v$, $y = 3\cosh u \sin v$, $z = 6\sinh u$

find a vector normal to the surface at the pt. for which $u = 1$, $v = \pi/3$

Sol. Given parametric eqs. of surface are

$$x = 2\cosh u \cos v$$
$$y = 3\cosh u \sin v$$
$$z = 6\sinh u$$

Let $P(x,y,z)$ be any pt. on given surface & $\underline{r}$ be its position vector then

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= (2\cosh u \cos v)\hat{i} + (3\cosh u \sin v)\hat{j} + (6\sinh u)\hat{k}$$

Now

$$\frac{\partial \underline{r}}{\partial u} = (2\sinh u \cos v)\hat{i} + (3\sinh u \sin v)\hat{j} + (6\cosh u)\hat{k}$$
$$= (2\sinh u . \cos\pi/3)\hat{i} + (3\sinh u \sin\pi/3)\hat{j} + (6\cosh u)\hat{k}$$
$$= (2\sinh u . \tfrac{1}{2})\hat{i} + (3\sinh u . \tfrac{\sqrt{3}}{2})\hat{j} + (6\cosh u)\hat{k}$$
$$\therefore \frac{\partial \underline{r}}{\partial u} = (\sinh u)\hat{i} + (\tfrac{3\sqrt{3}}{2}\sinh u)\hat{j} + (6\cosh u)\hat{k} \quad \text{at } v = \pi/3$$

Now

$$\frac{\partial \underline{r}}{\partial v} = (-2\cosh u \sin v)\hat{i} + (3\cosh u \cos v)\hat{j} + 0.\hat{k}$$
$$= (-2\cosh u . \sin\pi/3)\hat{i} + (3\cosh u \cos\pi/3)\hat{j} + 0\hat{k}$$
$$= (-2\cosh u . \tfrac{\sqrt{3}}{2})\hat{i} + (3\cosh u . \tfrac{1}{2})\hat{j} + 0\hat{k}$$
$$\frac{\partial \underline{r}}{\partial v} = (-\sqrt{3}\cosh u)\hat{i} + (\tfrac{3}{2}\cosh u)\hat{j} + 0\hat{k} \quad \text{at } v = \pi/3$$

Then the vector normal to the given surface is $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$

Now

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sinh u & \frac{3\sqrt{3}}{2}\sinh u & 6\cosh u \\ -\sqrt{3}\cosh u & \frac{3}{2}\cosh u & 0 \end{vmatrix}$$

$$= (0 - 9\cosh^2 u)\hat{i} - (0 + 6\sqrt{3}\cosh^2 u)\hat{j} + \left(\tfrac{3}{2}\sinh u\cosh u + \tfrac{9}{2}\sinh u\cosh u\right)\hat{k}$$

$$= -9\cosh^2 u\,\hat{i} + 6\sqrt{3}\cosh^2 u\,\hat{j} + 6\sinh u\cosh u\,\hat{k}$$

$$= \left[-9\cosh^2 u,\ 6\sqrt{3}\cosh^2 u,\ 3\sinh 2u\right] \quad \text{where } u = 1$$

is the req. normal vector to given surface.

Q20 For the surface defined by

$x = (3+\cos\phi)\cos\theta$

$y = (3+\cos\phi)\sin\theta$

$z = \sin\phi \qquad 0 \le \theta < 2\pi,\ - \ldots < q$

Show that the parametric curves for which ϕ is const. & θ varies are circles in planes parallel to xy-plane. Also show that the parametric curves for which θ is const. & ϕ varies lie in planes through z-axis. Find a vector normal to this surface at pt. for which $\theta = \pi/4$, $\phi = \frac{2\pi}{3}$

Sol. Given surface is

$x = (3+\cos\phi)\cos\theta$ —— ①

$y = (3+\cos\phi)\sin\theta$ —— ②

$z = \sin\phi$ —— ③

First we consider the parametric curves

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for which ϕ is Constt.

Sq. eqs. ① & ② & adding

$$x^2+y^2 = (3+\cos\phi)^2\left[\cos^2\theta+\sin^2\theta\right]$$

$$\boxed{x^2+y^2 = (3+\cos\phi)^2}$$

which are circles in planes $\parallel$ to xy-plane

Now we Consider the parametric curves for which θ is Constt.

from ① & ②

$$\frac{x}{\cos\theta} = 3+\cos\phi$$

$$\frac{y}{\sin\theta} = (3+\cos\phi)$$

subt. we get

$$\frac{x}{\cos\theta} - \frac{y}{\sin\theta} = c$$

$$x\sin\theta - y\cos\theta = c$$

which are planes through z-axis

Let $P(x,y,z)$ be any pt. on the surface & let $\underline{r}$ be its p.v. then

$$\underline{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$\underline{r} = (3+\cos\phi)\cos\theta\,\hat{i}+(3+\cos\phi)\sin\theta\,\hat{j}+\sin\phi\,\hat{k}$$

Now

$$\frac{\partial r}{\partial\theta} = -(3+\cos\phi)\sin\theta\,\hat{i}+(3+\cos\phi)\cos\theta\,\hat{j}+0\hat{k}$$

Put $\theta = \frac{\pi}{4}$, $\phi = 2\pi/3$

$$\frac{\partial r}{\partial\theta} = -(3+\cos 2\pi/3)\sin\pi/4\,\hat{i}+(3+\cos 2\pi/3)\cos\pi/4\,\hat{j}+0\hat{k}$$

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$$\frac{\partial r}{\partial \theta} = -\left(3-\frac{1}{2}\right)\frac{1}{\sqrt{2}}\hat{i} + \left(3-\frac{1}{2}\right)\frac{1}{\sqrt{2}}\hat{j} + 0\hat{k}$$

$$= -\left(\frac{5}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\hat{i} + \left(\frac{5}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\hat{j} + 0\hat{k}$$

$$\frac{\partial r}{\partial \theta} = -\frac{5}{2\sqrt{2}}\hat{i} + \frac{5}{2\sqrt{2}}\hat{j} + 0\hat{k} \quad \text{at } \theta = \pi/4,\ \phi = \frac{2\pi}{3}$$

Now

$$\frac{\partial r}{\partial \phi} = -\sin\phi\cos\theta\,\hat{i} - \sin\phi\sin\theta\,\hat{j} + \cos\phi\,\hat{k}$$

Put $\theta = \pi/4$, $\phi = 2\pi/3$

$$\frac{\partial r}{\partial \phi} = -\sin\frac{2\pi}{3}\cos\frac{\pi}{4}\,\hat{i} - \sin\frac{2\pi}{3}\sin\frac{\pi}{4}\,\hat{j} + \cos\frac{2\pi}{3}\,\hat{k}$$

$$= -\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\,\hat{i} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\,\hat{j} - \frac{1}{2}\,\hat{k}$$

$$\frac{\partial r}{\partial \phi} = -\frac{\sqrt{3}}{2\sqrt{2}}\hat{i} - \frac{\sqrt{3}}{2\sqrt{2}}\hat{j} - \frac{1}{2}\hat{k} \quad \text{at } \theta = \frac{\pi}{4},\ \phi = \frac{2\pi}{3}$$

Therefore the vector normal to the surface is

$$\frac{\partial r}{\partial \phi} \times \frac{\partial r}{\partial \theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2} \\ -\frac{5}{2\sqrt{2}} & \frac{5}{2\sqrt{2}} & 0 \end{vmatrix}$$

$$= \left(0 + \frac{5}{4\sqrt{2}}\right)\hat{i} - \left(0 - \frac{5}{4\sqrt{2}}\right)\hat{j} + \left(-\frac{5\sqrt{3}}{8} - \frac{5\sqrt{3}}{8}\right)\hat{k}$$

$$= \frac{5}{4\sqrt{2}}\hat{i} + \frac{5}{4\sqrt{2}}\hat{j} - \frac{10\sqrt{3}}{8}\hat{k}$$

$$= \frac{5}{4\sqrt{2}}\hat{i} + \frac{5}{4\sqrt{2}}\hat{j} - \frac{5\sqrt{3}}{4}\hat{k}$$

is req. vector normal to given surface.