Directional derivative.

Let $u = f(x,y,z)$ be a given surface & P be any pt. on u. Let Δs be the displacement of P in some specified direction & let Δu be the corresponding change in u then

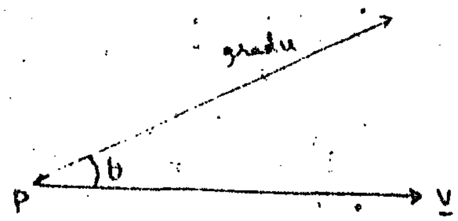
$\lim_{\Delta s \to 0} \frac{\Delta u}{\Delta s}$ if it exists is called directional derivative of u at pt. P in the specified direction & is denoted by $\frac{du}{ds}$.

Note Let the displacement $\Delta s = [\Delta x, \Delta y, \Delta z]$ is in direction of vector $\underline{V} = [a, b, c]$ then the directional derivative of $u = f(x,y,z)$ at pt. P in direction of vector V is

$$\boxed{\frac{du}{ds} = \frac{\text{grad}\, u \cdot \underline{V}}{|\underline{V}|}}$$ where $\text{grad}\, u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}$ & $\underline{V} = \overrightarrow{PO}$

or $$\frac{du}{ds} = \frac{|\text{grad}\, u|\,|\underline{V}|\cos\theta}{|\underline{V}|}$$

or $$\boxed{\frac{du}{ds} = |\text{grad}\, u|_P \cos\theta}$$



Note As $\frac{du}{ds} = |\text{grad}\, u|_P \cos\theta$

(i) If $\theta = 0$ then $\cos 0 = 1$

Hence $\frac{du}{ds} = |\text{grad}\, u|_P$

So directional derivative $\frac{du}{ds}$ is greatest in direction of grad u

(ii) If $\theta = 90^\circ$ then $\cos 90^\circ = 0$

Hence $\frac{du}{ds} = 0$

So direction derivative is zero in direction $\perp$ to grad(u).

Exercise No. 9.4

In problems 1–3, find the rate of change of u at the given pt. & in the given direction.

Q1 $u = 2xy - \dfrac{y}{x}$; $(1,2)$; $[2,-3,0]$

Sol. Given surface is

$$u = 2xy - \frac{y}{x}$$

Then

$$\left.\begin{aligned}
\frac{\partial u}{\partial x} &= 2y + \frac{y}{x^2} = 2(2) + \frac{2}{1} = 6 \\
\frac{\partial u}{\partial y} &= 2x - \frac{1}{x} = 2(1) - \frac{1}{1} = 1 \\
\frac{\partial u}{\partial z} &= 0
\end{aligned}\right\} \text{ at } (1,2,0)$$

So

$$\text{grad}\, u = \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right] = [6, 1, 0]$$

Here $\underline{v} = [2,-3,0]$

So req. rate of change of u is

$$\frac{du}{ds} = \frac{\text{grad}\, u \cdot \underline{v}}{|\underline{v}|}$$

$$= \frac{[6,1,0]\cdot[2,-3,0]}{\sqrt{4+9+0}}$$

$$= \frac{(6)(2) + (1)(-3) + (0)(0)}{\sqrt{13}}$$

$$= \frac{12-3}{\sqrt{13}}$$

$$\frac{du}{ds} = \frac{9}{\sqrt{13}}$$

Q2 $u = ye^{-x}(x^2+y^2+z^2+1)$; $(0,0,0)$; $[2,1,2]$

Sol: Given surface is

$$u = ye^{-x}(x^2+y^2+z^2+1)$$

then

$$\frac{\partial u}{\partial x} = y\left[e^{-x}(2x)+(x^2+y^2+z^2+1)\cdot e^{-x}(-1)\right]$$

$$= ye^{-x}\left[2x-x^2-y^2-z^2-1\right] = 0$$

$$\frac{\partial u}{\partial y} = e^{-x}\left[y(2y)+(x^2+y^2+z^2+1)\cdot 1\right]$$

$$= e^{-x}\left[2y^2+x^2+y^2+z^2+1\right]$$

$$= e^{-x}\left[x^2+3y^2+z^2+1\right]$$

$$= e^{0}\left[0+0+0+1\right]$$

$$= 1$$

& $\frac{\partial u}{\partial z} = ye^{-x}(2z) = 0$

As $\text{grad}\,u = \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right]$

$$\text{grad}\,u = [0,1,0]$$

Here $\underline{v} = [2,1,2]$

then the req. rate of change of u is

$$\frac{du}{ds} = \frac{\text{grad}\,u \cdot \underline{v}}{|\underline{v}|}$$

$$= \frac{[0,1,0]\cdot[2,1,2]}{\sqrt{4+1+4}}$$

$$= \frac{0+1+0}{3}$$

$$= \frac{1}{3}$$

Q3 $u = \sinh(x+y) + \cosh z$; $(1,0,1)$; $[-2,2,-1]$

Sol. Given eq. of surface is

$$u = \sinh(x+y) + \cosh z$$

then

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \cosh(x+y) = \cosh(1+0) = \cosh(1) \\ \frac{\partial u}{\partial y} &= \cosh(x+y) = \cosh(1+0) = \cosh(1) \\ \frac{\partial u}{\partial z} &= \sinh z = \sinh(1) \end{aligned}\right\} \text{ at } (1,0,1)$$

Then

$$\text{grad } u = \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right]$$

$$= [\cosh(1), \cosh(1), \sinh(1)]$$

Here $\underline{v} = [-2, 2, -1]$

then the req. rate of change of u is

$$\frac{du}{ds} = \frac{\text{grad}(u) \cdot \underline{v}}{|\underline{v}|}$$

$$= \frac{[\cosh(1), \cosh(1), \sinh(1)] \cdot [-2, 2, -1]}{\sqrt{4+4+1}}$$

$$= \frac{-2\cosh(1) + 2\cosh(1) - \sinh(1)}{\sqrt{9}}$$

$$= -\frac{1}{3}\sinh(1)$$

$$= -\frac{1}{3}\left(\frac{e^{1} - e^{-1}}{2}\right)$$

$$= -\frac{1}{6}\left(e - \frac{1}{e}\right)$$

$$= -\frac{1}{6}\left(\frac{e^2 - 1}{e}\right)$$

$$= \frac{1 - e^2}{6e}$$

Q4 Let $u = x^2+y^2$, find the direction of the greatest rate of change of u at (a,b) & the magnitude of this greatest rate of change. Find the direction of no change at (a,b).

Sol: Given surface is

$$u = x^2 + y^2$$

then
$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2x = 2a \\ \frac{\partial u}{\partial y} &= 2y = 2b \\ \frac{\partial u}{\partial z} &= 0 \end{aligned}\right\} \text{ at } (a,b)$$

Then $\text{grad } u = \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right]$

or $\text{grad } u = [2a, 2b, 0]$

We know that directional derivative is the greatest in the direction of gradient itself.

So $\text{grad } u = [2a, 2b, 0]$ is the direction of greatest rate of change of u & its magnitude is

$$|\text{grad } u| = \sqrt{4a^2+4b^2+0^2}$$

$$= 2\sqrt{a^2+b^2}$$

Now the direction of no change of u at (a,b) is the vector perp. to $\text{grad}(u) = [2a, 2b, 0]$ which is $[-2b, 2a, 0]$

Q5 If $u = \tan^{-1}(y/x)$, find the direction of greatest rate of change of u at (a,b) & the magnitude of the greatest rate of change. Find also the direction of no change at (a,b).

Sol: Given surface is

$u = \tan^{-1}(y/x)$

then

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = \frac{x^2}{x^2+y^2} \cdot -\frac{y}{x^2}$$

$$= \frac{-y}{x^2+y^2} = -\frac{b}{a^2+b^2}.$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{a}{a^2+b^2}$$

& $\frac{\partial u}{\partial z} = 0$ at pt. (a, b)

Now $\text{grad}\, u = \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right]$

$$= \left[-\frac{b}{a^2+b^2}, \frac{a}{a^2+b^2}, 0\right]$$

Now direction of greatest change of u at (a, b) is

$$= \text{grad}(u)$$

$$= \left[-\frac{b}{a^2+b^2}, \frac{a}{a^2+b^2}, 0\right]$$

& magnitude of greatest change of u is

$$|\text{grad}\, u| = \sqrt{\frac{b^2}{(a^2+b^2)^2} + \frac{a^2}{(a^2+b^2)^2} + 0^2}$$

$$= \sqrt{\frac{b^2}{(a^2+b^2)^2} + \frac{a^2}{(a^2+b^2)^2}}$$

$$= \sqrt{\frac{(a^2+b^2)}{(a^2+b^2)^2}}$$

$$= \frac{1}{\sqrt{a^2+b^2}}$$

Now direction of no change of u at (a, b) is a vector $\perp$ to grad u which is $[a, b, 0]$

Q6 The temperature distribution for the semi circular plate $x^2+y^2 \le 1$, $y \geq 0$ is given by the formula $T = 3x^2y - y^3 + 27z$ under certain conditions. Find $\frac{dT}{ds}$ at $A(0, \frac{1}{2})$ in the direction of y-axis. Also find (i) $\frac{dT}{ds}$ at A in the direction of $[1, -2]$.
(ii) The magnitude of greatest rate of change at A.
(iii) The magnitude of the greatest rate of change
(IV) The direction of the isothermal through A (the direction of zero rate of change at A).

Sol. Given

$$T = 3x^2y - y^3 + 27z$$

then
$$\left.\begin{aligned} \frac{\partial T}{\partial x} &= 6xy = 0 \\ \frac{\partial T}{\partial y} &= 3x^2 - 3y^2 = 0 - 3(1/4) = -\frac{3}{4} \end{aligned}\right\} \text{ at } (0, \tfrac{1}{2})$$

Now $\text{grad } T = \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right]$

or $\text{grad } T = [0, -3/4]$

Now unit vector along y-axis is $\hat{j} = [0, 1]$

Then $\frac{dT}{ds} = \text{grad } T \cdot \hat{j}$

$$= [0, -\tfrac{3}{4}] \cdot [0, 1]$$

$$= 0 \cdot 0 + (-\tfrac{3}{4})(1)$$

$$\boxed{\frac{dT}{ds} = -\frac{3}{4}} \text{ at } (0, \tfrac{1}{2})$$

(i) Here $\underline{V} = [1, -2]$

then $\frac{dT}{ds}$ in direction of $\underline{V}$ $= \dfrac{\text{grad } T \cdot \underline{V}}{|\underline{V}|}$

$$= \frac{[0, -3/4] \cdot [1, -2]}{\sqrt{1+4}}$$

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$$= \frac{0 + (-3/4)(-2)}{\sqrt{5}}$$

So $\frac{dT}{ds}$ in direction of $\underline{V} = \frac{3}{2\sqrt{5}}$

(ii) Now the direction of greatest rate of change $=$ grad T

$$= \left[0, -\frac{3}{4}\right]$$

(iii) Magnitude of greatest rate of change $= |\text{grad}\,T|$

$$= \sqrt{0^2 + \frac{9}{16}}$$

$$= \sqrt{\frac{9}{16}}$$

$$= \frac{3}{4}$$

(iv) Direction of the isothermal through pt. A $=$ Direction $\perp$ to grad T

$$= \hat{i} = [1, 0]$$

Q7 Repeat problem 6 with $T = 4x^3y - 4xy^3 + 27z$ and $A\left(\frac{1}{2}, \frac{1}{2}\right)$

Sol. Given

$$T = 4x^3y - 4xy^3 + 27z$$

Then

$$\left.\begin{aligned} \frac{\partial T}{\partial x} &= 12x^2y - 4y^3 = 12\left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right) - 4\cdot\tfrac{1}{8} = \tfrac{3}{2} - \tfrac{1}{2} = 1 \\ \frac{\partial T}{\partial y} &= 4x^3 - 12xy^2 = 4\left(\tfrac{1}{8}\right) - 12\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right) = \tfrac{1}{2} - \tfrac{3}{2} = -1 \end{aligned}\right\} \text{ at } A\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Now grad T $= \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right]$

$$= [1, -1]$$

Now a unit vector in direction of y-axis.

is $\hat{j} = [0, 1]$

Then $\frac{dT}{ds}$ in direction of y-axis $= \text{grad}T \cdot \hat{j}$

$= [1, -1] \cdot [0, 1]$

$= 1.0 + (-1)(1)$

$= 0 - 1$

$= -1$

(i) Here $\underline{v} = [1, -2]$

then

$$\frac{dT}{ds} \text{ in direction of } \underline{v} = \frac{\text{grad}T \cdot \underline{v}}{|\underline{v}|}$$

$$= \frac{[1, -1] \cdot [1, -2]}{\sqrt{1+4}}$$

$$= \frac{1.1 + (-1)(-2)}{\sqrt{5}}$$

$$= \frac{1+2}{\sqrt{5}}$$

$$= \frac{3}{\sqrt{5}}$$

(ii) The direction of greatest rate of Change $= \text{grad}T$

$= [1, -1]$

(iii) Magnitude of greatest rate of change $= |\text{grad}T|$

$= \sqrt{1+1}$

$= \sqrt{2}$

(iv) Direction of isothermal through A = Direction of no change at A

= Direction $\perp$ to gradT

$= [1, 1]$