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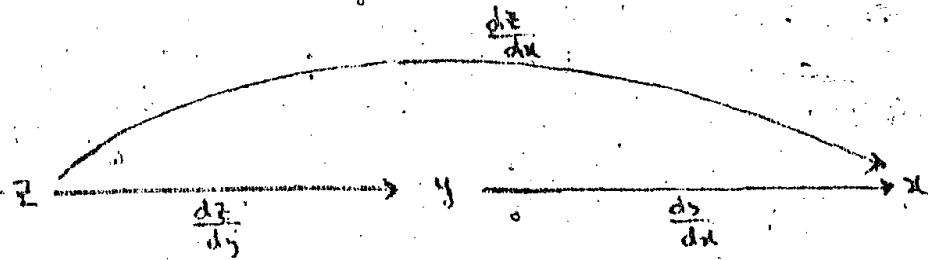


Differentiation of Composite functions

Chain rules:

① If z is a function of y & y is a function of x , then z is a function of x only, then

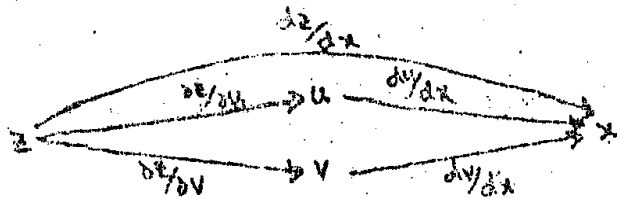
$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$



② If $z = f(u, v)$ & $u = u(x)$, $v = v(x)$

Then z is a fn. of single variable x so

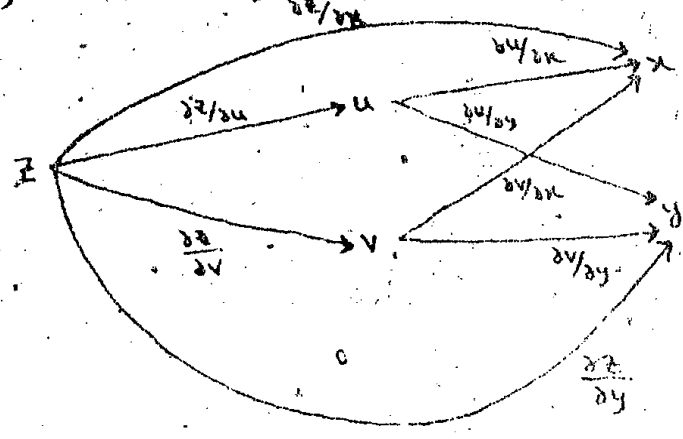
$$\frac{dz}{dx} = \frac{\partial z}{\partial u} \cdot \frac{du}{dx} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dx}$$



③ If $z = f(u, v)$ & $u = u(x, y)$, $v = v(x, y)$ then z is a function of x & y , so

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$$

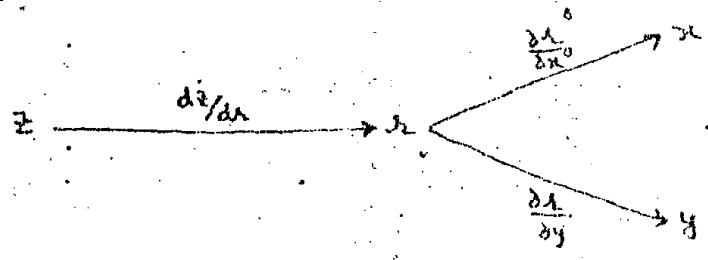
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$



④ If $z = f(r)$ & $r = (x, y)$. Then

$$\frac{\partial z}{\partial x} = \frac{dz}{dr} \frac{\partial r}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{dr} \frac{\partial r}{\partial y}$$



Q1. If $u = x - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$ then
find $\frac{\partial u}{\partial r}$ + $\frac{\partial u}{\partial s}$

Soln. Given

$$u = x - y^2, \quad x = 2r - 3s + 4, \quad y = -r + 8s - 5$$

Hence u is actually a fn. of $r + s$

By chain rule..

$$\left. \begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} \\ + \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} \end{aligned} \right\}$$

$$\left. \begin{aligned} \frac{\partial u}{\partial r} &= (1)(2) + (-2y)(-1) \\ \frac{\partial u}{\partial s} &= (1)(-3) + (-2y)(8) \end{aligned} \right\}$$

$$\left. \begin{aligned} \text{or } \frac{\partial u}{\partial r} &= 2 + 2y \\ \frac{\partial u}{\partial s} &= -3 - 16y \end{aligned} \right\}$$

$$\left. \begin{aligned} \text{or } \frac{\partial u}{\partial r} &= 2(1+y) \\ \frac{\partial u}{\partial s} &= -(3+16y) \end{aligned} \right\}$$

Q2. If $z = \frac{e^{xy}}{x}$, $x = u^2 - v$, $y = e^v$

then find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$

Soln. Given $z = \frac{e^{xy}}{x}$, $x = u^2 - v$, $y = e^v$

So z is actually a fn. of $u + v$

Hence by chain rule

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$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\text{as } z = \frac{\cos y}{x} \quad , \quad x = u^2 - v \quad , \quad y = e^v$$

$$\frac{\partial z}{\partial x} = -\frac{\cos y}{x^2} \quad , \quad \frac{\partial x}{\partial u} = 2u \quad , \quad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{\sin y}{x} \quad , \quad \frac{\partial x}{\partial v} = -1 \quad , \quad \frac{\partial y}{\partial v} = e^v$$

So

$$\frac{\partial z}{\partial u} = -\frac{\cos y}{x^2} \cdot 2u + (-\frac{\sin y}{x}) \cdot 0$$

$$\boxed{\frac{\partial z}{\partial u} = -\frac{2u \cos y}{x^2}}$$

$$\frac{\partial z}{\partial v} = -\frac{\cos y}{x^2} \cdot (-1) + (-\frac{\sin y}{x}) \cdot e^v$$

$$= \frac{\cos y}{x^2} - \frac{\sin y \cdot e^v}{x}$$

$$= \frac{\cos y - x e^v \sin y}{x^2}$$

$$\frac{\partial z}{\partial v} = \frac{1}{x^2} [\cos y - x y \sin y]$$

Note If $f(x, y) = 0$ then to find $\frac{dy}{dx}$, use use

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}$$

$$\text{or } \frac{dy}{dx} = -\frac{f_x}{f_y}$$

Find $\frac{dy}{dx}$ (Problems 3-6):

Q3 $\sin(xy) - e^{-xy} - x^2y = 0$

Sol.

let $f(x,y) = \sin(xy) - e^{-xy} - x^2y$

Diff. partially w.r.t. x & y

$$f_x = \cos(xy) \cdot y - e^{-xy} \cdot y = 2xy$$

$$f_x = y \cos(xy) - y e^{-xy} - 2xy$$

$$f_y = \cos(xy) \cdot x - e^{-xy} \cdot x - x^2$$

$$\text{or } f_y = x \cos(xy) - x e^{-xy} - x^2$$

Now

$$\frac{dy}{dx} = \frac{f_x}{f_y}$$

$$\frac{dy}{dx} = \frac{y \cos(xy) - y e^{-xy} - 2xy}{x \cos(xy) - x e^{-xy} - x^2}$$

$$= \frac{y(\cos(xy) - e^{-xy} - 2x)}{x(x + e^{-xy} - \cos(xy))}$$

Q4 $3(x^2+y^2)^2 = 25(x^2-y^2)$

Sol.

Given

$$3(x^2+y^2)^2 = 25(x^2-y^2)$$

$$\text{or } 3(x^2+y^2)^2 - 25(x^2-y^2) = 0$$

let $f(x,y) = 3(x^2+y^2)^2 - 25(x^2-y^2)$

Diff. partially w.r.t. x & y

$$f_x = 6(x^2+y^2) \cdot 2x - 25(2x)$$

$$= 12x(x^2+y^2) - 50x$$

$$f_y = 6(x^2 + y^2) \cdot 2y - 25(-2y)$$

$$f_y = 12y(x^2 + y^2) + 50y$$

Now

$$\frac{dy}{dx} = - \frac{f_x}{f_y}$$

$$= - \frac{12x(x^2 + y^2) - 50x}{12y(x^2 + y^2) + 50y}$$

$$= - \frac{-x(25 - 6(x^2 + y^2))}{y(6(x^2 + y^2) + 25)}$$

$$\frac{dy}{dx} = \frac{25x - 6x(x^2 + y^2)}{25y + 6y(x^2 + y^2)}$$

Q5 $f(x, y) = x^y - y^x = 0$

Sol. Here $f(x, y) = x^y - y^x$

Diff. partially w.r.t. x & y

$$f_x = yx^{y-1} - y^x \ln y$$

$$f_y = x^y \ln x - xy^{x-1}$$

Now $\frac{dy}{dx} = - \frac{f_x}{f_y}$

$$= - \frac{yx^{y-1} - y^x \ln y}{x^y \ln x - xy^{x-1}}$$

$$= \frac{y^x \ln y - yx^{y-1}}{x^y \ln x - xy^{x-1}}$$

$$= \frac{y^x \ln y - y \cdot \frac{y^x}{x}}{x^y \ln x - x \cdot \frac{y^x}{y}}$$

$$= \frac{y^x \ln y - \frac{y^{x+1}}{x}}{x^y \ln x - \frac{xy^x}{y}}$$

$$\begin{aligned}
 &= \frac{x^y \ln y - \frac{y}{x} \cdot x^y}{x^y \ln x - \frac{x}{y} \cdot x^y} \quad (\because x^y = y^x) \\
 &= \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} \\
 &= \frac{x \ln y - y}{y \ln x - x} \\
 &= \frac{y(x \ln y - y)}{x(y \ln x - x)} \\
 \frac{dy}{dx} &= \frac{y(y - x \ln y)}{x(x - y \ln x)}
 \end{aligned}$$

Q6 $(\tan x)^y + y^{\cot x} = a$

Sol: Given

$$(\tan x)^y + y^{\cot x} = a$$

$$\text{or } (\tan x)^y + y^{\cot x} - a = 0$$

Here $f(x, y) = (\tan x)^y + y^{\cot x} - a$

Diff. partially w.r.t. x & y

$$f_x = y(\tan x)^{y-1} \cdot \sec^2 x + y^{\cot x} \cdot \ln y - \cot x \cdot y^{\cot x - 1}$$

$$f_x = y \sec^2 x (\tan x)^{y-1} - \ln y \cdot \cot x \cdot y^{\cot x - 1}$$

$$\& f_y = (\tan x)^y \cdot \ln \tan x + \cot x \cdot y^{\cot x - 1}$$

Now $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$\text{or } \frac{dy}{dx} = -\frac{y \sec^2 x (\tan x)^{y-1} - \ln y \cdot \cot x \cdot y^{\cot x - 1}}{(\tan x)^y \cdot \ln \tan x + \cot x \cdot y^{\cot x - 1}} \quad \text{Ans.}$$

Q7 If $F(x, y, z) = 0$, find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

Sol. Let $w = F(x, y, z)$

then its differential is

$$dw = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$$

$$\text{But } w = 0$$

$$\Rightarrow dw = 0$$

Hence above eq. becomes

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0 \quad \text{--- (1)}$$

But we know that when z is defined explicitly as a function of two independent variables x & y

$$\text{i.e., } z = f(x, y)$$

then its differential is

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Put in (1)

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} \left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \right) = 0$$

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} \right) dx + \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} \right) dy = 0 \quad \text{--- (2)}$$

To find $\frac{\partial z}{\partial x}$, we keep y const. so $dy = 0$

Hence eq. (2) becomes

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} \right) dx = 0$$

$$\Rightarrow dx \neq 0$$

$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = - \frac{\partial F / \partial x}{\partial F / \partial z}}$$

Now to find $\frac{\partial z}{\partial y}$ we keep x const. so $dx = 0$ 35

Hence eq. (1) becomes

$$\left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} \right) dy = 0$$

$$\therefore dy \neq 0$$

$$\Rightarrow \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} = 0$$

$$\text{or } \boxed{\frac{\partial z}{\partial y} = - \frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}}$$

Q8 If $f(x, y) = 0$ & $\phi(y, z) = 0$, show that

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$

Sol. Given that $f(x, y) = 0$

$$\Rightarrow \frac{dy}{dx} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \text{--- (1)}$$

Also given

$$\phi(y, z) = 0$$

$$\Rightarrow \frac{dz}{dy} = - \frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}} \quad \text{--- (2)}$$

Multiplying (1) & (2)

$$\frac{dy}{dx} \cdot \frac{dz}{dy} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \cdot - \frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$$

$$\frac{dz}{dx} = \frac{\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}}{\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z}}$$

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y} \quad \text{Ans.}$$

Q9 If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$, show that

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$$\frac{dy}{dx} = \frac{a}{(1-x^2)^{3/2}}$$

Sol. Given

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$$

$$\text{or } x\sqrt{1-y^2} + y\sqrt{1-x^2} - a = 0$$

$$\text{Here } f(x,y) = x\sqrt{1-y^2} + y\sqrt{1-x^2} - a$$

Diff. partially w.r.t. x & y

$$f_x = \sqrt{1-y^2} + y \cdot \frac{1}{2\sqrt{1-x^2}} (-2x)$$

$$= \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

$$f_x = \frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-x^2}}$$

$$\text{Now } f_y = x \cdot \frac{1}{2\sqrt{1-y^2}} (-2y) + \sqrt{1-x^2}$$

$$= -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$= \frac{-xy + \sqrt{1-x^2}\sqrt{1-y^2}}{\sqrt{1-y^2}}$$

$$f_y = \frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-y^2}}$$

Now

$$\frac{dy}{dx} = \frac{f_x}{f_y} = \frac{\frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-x^2}}}{\frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-y^2}}}$$

$$\frac{dy}{dx} = - \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

Diff: w.r.t. x

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2} \left(\frac{1}{2\sqrt{1-y^2}} \cdot -2y \frac{dy}{dx} \right) - \sqrt{1-y^2} \left(\frac{1}{2\sqrt{1-x^2}} (-2x) \right)}{(1-x^2)}$$

$$= \frac{\sqrt{1-x^2} \cdot \frac{-y}{\sqrt{1-y^2}} - \sqrt{1-y^2} \cdot \frac{-x}{\sqrt{1-x^2}}}{(1-x^2)}$$

$$= \frac{y + \frac{x\sqrt{1-y^2}}{\sqrt{1-x^2}}}{(1-x^2)}$$

$$= \frac{y\sqrt{1-x^2} + x\sqrt{1-y^2}}{(1-x^2)\sqrt{1-x^2}}$$

$$\frac{dy}{dx^2} = \frac{a}{(1-x^2)^{3/2}}$$

Q10 If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that

$$\frac{dy}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^2}$$

Sol: Given that

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here $f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

Then $f_x = 2ax + 2hy + 2g$

$f_y = 2by + 2hx + 2f$

Now $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$= -\frac{2ax + 2hy + 2g}{2by + 2hx + 2f}$$

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}$$

Diff. w.r.t. x

$$\frac{d^2y}{dx^2} = -\frac{(hx + by + f) \cdot (a - h \frac{dy}{dx}) - (ax + hy + g) (h + b \frac{dy}{dx})}{(hx + by + f)^2}$$

$$= -\frac{\frac{dy}{dx} [h^2x + hb^2y + hf - abx - h^2y - gb] + [ah^2x + aby + af - ah^2x - h^2y - gh]}{(hx + by + f)^2}$$

$$= -\frac{\left(-\frac{ax + hy + g}{hx + by + f}\right) [h^2x + hf - abx - gb] + [aby + af - h^2y - gh]}{(hx + by + f)^2}$$

$$= \frac{(ax + hy + g)(h^2x + hf - abx - gb) - (aby + af - h^2y - gh)(hx + by + f)}{(hx + by + f)^3}$$

$$= \frac{ah^2x^2 + ah^2fy - abx^2 - abgx + h^3xy + h^2fy - abhxy - gh^2y + h^2gx + ghf - abgx - g^2b}{(hx + by + f)^3}$$

$$= \frac{-abhxy - ah^2y - abfy - ah^2x - abfy - ah^2x^2 + h^3xy + h^2by + h^2fy + gh^2x + gh^2y + ghf}{(hx + by + f)^3}$$

$$= \frac{ah^2x^2 - abx^2 - 2abgx + 2h^3xy + 2h^2fy - 2abhxy + 2h^2gx + 2ghf - g^2b - ah^2y - 2abfy - ah^2x - h^2by^2}{(hx + by + f)^3}$$

$$\frac{dy}{dx} = \frac{h^2(ax^2 + 2hxy + 2fy + 2gx + by^2) - ab(ax^2 + 2gx + 2hxy + by^2 + 2fy)}{(hx + by + f)^2} \quad \frac{39}{+2ghf - g^2b - af^2}$$

$$= \frac{h^2(-c) - ab(-c) + 2ghf - g^2b - af^2}{(hx + by + f)^2}$$

$$\frac{dy}{dx} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^2}$$

Q11 Find $\frac{dy}{dx}$: if $x^3 + y^3 = 3axy$

Sol: Given

$$x^3 + y^3 = 3axy$$

$$\text{or } x^3 + y^3 - 3axy = 0$$

$$\text{Let } f(x, y) = x^3 + y^3 - 3axy$$

$$\text{Then } f_x = 3x^2 - 3ay$$

$$f_y = 3y^2 - 3ax$$

$$\text{Now } \frac{dy}{dx} = \frac{f_x}{f_y}$$

$$= \frac{3x^2 - 3ay}{3y^2 - 3ax}$$

$$= \frac{x^2 - ay}{y^2 - ax}$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{(y^2 - ax) \left[a \frac{dy}{dx} - 2x \right] - (ay - x^2) \left[2y \frac{dy}{dx} - a \right]}{(y^2 - ax)^2}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{ay \frac{dz}{dx} - 2xy - a^2 x \frac{dz}{dx} + 2ax^2 - 2ay \frac{dz}{dx} + a^2 y + 2xy \frac{dy}{dx} - ax^2}{(y^2 - ax)^2} \\
 &= \frac{\frac{dy}{dx} (ay^2 - a^2 x - 2ay^2 + 2x^2) - (2xy^2 - 2ax^2 - a^2 y + ax^2)}{(y^2 - ax)^2} \\
 &= \frac{\left(\frac{ay - x^2}{y^2 - ax}\right) (2x^2 - ay^2 - a^2 x) - (2xy^2 - ax^2 - a^2 y)}{(y^2 - ax)^2} \\
 &= \frac{(ay - x^2)(2x^2 - ay^2 - a^2 x) - (2xy^2 - ax^2 - a^2 y)(y^2 - ax)}{(y^2 - ax)^3} \\
 &= \frac{2ax^2y - a^2y^3 - a^2xy - 2x^4y + ax^2y^2 + a^2x^3 - (2xy^4 + 2ax^2y^2 - a^2xy + a^2x^3 - a^2y^3 + a^2xy)}{(y^2 - ax)^3} \\
 &= \frac{3ax^2y^2 - a^2y^3 - a^2xy - 2x^4y + a^2x^3 - (2xy^4 - 3ax^2y^2 + a^2x^3 - a^2y^3 + a^2xy)}{(y^2 - ax)^3} \\
 &= \frac{3ax^2y^2 - a^2y^3 - a^2xy - 2x^4y + a^2x^3 - 2xy^4 + 3ax^2y^2 - a^2x^3 + a^2y^3 - a^2xy}{(y^2 - ax)^3} \\
 &= \frac{6ax^2y^2 - 2a^2xy - 2xy(x^3 + y^3)}{(y^2 - ax)^3} \\
 &= \frac{6ax^2y^2 - 2a^2xy - 2xy(3axy)}{(y^2 - ax)^3} \\
 &= \frac{6ax^2y^2 - 2a^2xy - 6ax^2y^2}{(y^2 - ax)^3} \\
 &= \frac{-2a^2xy}{(y^2 - ax)^3} \\
 \frac{dy}{dx} &= \frac{2a^2xy}{(ax - y^2)^3}
 \end{aligned}$$