# Exercise 8.10

Derive an eq. of the Cylinder from definition with given directrix & elements parallel to the given vector (Problems 1–3).

**Q1** $x^2+y^2=9$ , $\vec{n}=[1,-2,1]$

**Sol.** The eq. of line L through $(x_1,y_1,0)$ & parallel to $\vec{n}$ is

$$\frac{x-x_1}{1}=\frac{y-y_1}{-2}=\frac{z}{1}=t\ (\text{say})$$

$\Rightarrow$ $x=x_1+t$ ——— (i)

$y=y_1-2t$ ——— (ii)

$z=t$ ——— (iii)

From (iii) $t=z$

Put in (i) & (ii)

$$\left.\begin{aligned} x&=x_1+z\\ y&=y_1-2z\end{aligned}\right]$$

or $$\left.\begin{aligned} x_1&=x-z\\ y_1&=y+2z\end{aligned}\right]$$

Since $(x_1,y_1,0)$ lies on $x^2+y^2=9$

So $x_1^2+y_1^2=9$

$(x-z)^2+(y+2z)^2=9$

$x^2-2xz+z^2+y^2+4yz+4z^2=9$

$\boxed{x^2+y^2+5z^2-2xz+4yz=9}$ is req. eq. of cylinder 172

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**Q2** $\quad x+z=4, \quad ; \quad \vec{n}=[0,2,-1]$

**Sol.** The eq. of line L through $(x_1, 0, z_1)$ & $\parallel$ to $\vec{n}$ is

$$\frac{x-x_1}{0}=\frac{y-0}{2}=\frac{z-z_1}{-1}=t \text{ (say)}$$

$$\Rightarrow x=x_1 \quad \text{---(i)}$$
$$y=2t \quad \text{---(ii)}$$
$$z=z_1-t \quad \text{---(iii)}$$

From (ii) $t=\frac{y}{2}$

Put in (i) & (iii)

$$\left.\begin{aligned} x&=x_1 \\ z&=z_1-\frac{y}{2}\end{aligned}\right)$$

or $$\left.\begin{aligned} x_1&=x \\ z_1&=z+\frac{y}{2}\end{aligned}\right)$$

Since $(x_1, 0, z_1)$ lies on $x+z=4$

So $x_1+z_1=4$

using values of $x_1$ & $z_1$

$$x+z+\frac{y}{2}=4$$

$$2x+z+y=8$$

$\boxed{2x+y+2z=8}$ is req. eq. of cylinder.

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Q3 $\frac{z^2}{4}+\frac{y^2}{9}=1$ ; $\vec{n}=[1,1,1]$ 

Sol. The eq. of line through $(0,y_1,z_1)$ & $\parallel$ to $\vec{n}$ is

$$\frac{x-0}{1}=\frac{y-y_1}{1}=\frac{z-z_1}{1}=t \text{ (say)}$$

$$\Rightarrow x=t \quad \text{——— (i)}$$
$$y=y_1+t \quad \text{——— (ii)}$$
$$z=z_1+t \quad \text{——— (iii)}$$

From (i) $t=x$

Put in (ii) & (iii)

$$\left.\begin{aligned} y &= y_1+x \\ z &= z_1+x \end{aligned}\right\}$$

or
$$\left.\begin{aligned} y_1 &= y-x \\ z_1 &= z-x \end{aligned}\right\}$$

Since $(0,y_1,z_1)$ lies on $\frac{z^2}{4}+\frac{y^2}{9}=1$

So $\frac{z_1^2}{4}+\frac{y_1^2}{9}=1$

Using values of $y_1$ & $z_1$

$$\frac{(z-x)^2}{4}+\frac{(y-x)^2}{9}=1$$

$$9(z-x)^2+4(y-x)^2=36$$

$$9(z^2+x^2-2zx)+4(y^2-2xy+x^2)=36$$

$$9z^2+9x^2-18xz+4y^2-8xy+4x^2=36$$

$$\boxed{13x^2+4y^2+9z^2-8xy-18xz-36=0}$$ is req. eq.

**Q12** Write an eq. of right circular cylinder[174] with radius 2 & centre at (3,0,5).

**Sol.** The eq. of circle with centre at (3,0,5) & radius 2 is

$$(x-3)^2+(z-5)^2=(2)^2$$

or $\boxed{(x-3)^2+(z-5)^2=4}$ is eq. of directrix

Which is also the eq. of right circular cylinder.

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**Q13** Write an eq. of the right elliptic cylinder whose directrix is in the xz-plane with foci $(0,\pm3,0)$ & major axis 8.

**Sol.** ~~Here~~ $2a=8$

$$\Rightarrow \boxed{a=4}$$

$$\& \quad c=3$$

For an ellipse

$$b^2=a^2-c^2$$

$$=(4)^2-(3)^2$$

$$=16-9$$

$$b^2=7$$

$$\boxed{b=\sqrt{7}}$$

The eq. of DTX which is an ellipse is

$$\frac{x^2}{16} + \frac{y^2}{7} = 1$$

or $\quad 7x^2 + 16y^2 = 112$

$$\boxed{7x^2 + 16y^2 - 112 = 0}$$

which is also eq. of right elliptic cylinder.

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Find an Eqⁿ of the Cone whose directrix & vertex are given (Problems 14–16):

**Q14** Directrix : $y^2 + z^2 = 1$ , $x = 2$ vertex $A(0,0,0)$

**Sol.** Let $P(x,y,z)$ be a pt. on the req. Cone & let line AP meets directrix at pt. $Q(x_1,y_1,z_1)$

Since pt. Q lies on directrix

i.e. $y_1^2 + z_1^2 = 1$ , $x_1 = 2$ —— (1)

Then eq. of line AQ is

$$\frac{x-0}{x_1-0} = \frac{y-0}{y_1-0} = \frac{z-0}{z_1-0}$$

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}$$

From (1) Put $x_1 = 2$

$$\frac{x}{2} = \frac{y}{y_1} , \quad \frac{x}{2} = \frac{z}{z_1}$$

or $\quad \dfrac{y_1}{y} = \dfrac{2}{x} , \quad \dfrac{z_1}{z} = \dfrac{2}{x}$

or $y_1 = \frac{2y}{x}$ , $z_1 = \frac{2z}{x}$

Put values in ①

$$\frac{4y^2}{x^2} + \frac{4z^2}{x^2} = 1$$

$$4y^2 + 4z^2 = x^2$$

$\boxed{4(y^2+z^2) = x^2}$ is req. eq. of Cone.



**Q15** Directrix $4x^2+(y-2)^2 = 4,\ z = 3$ , Vertex $A(0,0,0)$

**Soln** Let $P(x,y,z)$ be any pt. on req. Cone & let line AP meets directrix at pt. $Q(x_1,y_1,z_1)$

Since $Q(x_1,y_1,z_1)$ lies on directrix

so $4x_1^2 + (y_1-2)^2 = 4$ , $z_1 = 3$ ——— ①

The eq. of line AQ is

$$\frac{x-0}{x_1-0} = \frac{y-0}{y_1-0} = \frac{z-0}{z_1-0}$$

or $\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}$

Put $z_1 = 3$

$$\frac{x}{x_1} = \frac{z}{3} , \quad \frac{y}{y_1} = \frac{z}{3}$$

$$\frac{x_1}{x} = \frac{3}{z} , \quad \frac{y_1}{y} = \frac{3}{z}$$

$\Rightarrow x_1 = \frac{3x}{z}$ , $y_1 = \frac{3y}{z}$

Put in ①

$$4\left(\frac{3x}{z}\right)^2 + \left(\frac{3y}{z}-2\right)^2 = 4$$

$$4\left(\frac{9x^2}{z^2}\right) + \left(\frac{3y-2z}{z}\right)^2 = 4$$

$$\frac{36x^2}{z^2} + \frac{9y^2-12yz+4z^2}{z^2} = 4$$

$$36x^2 + 9y^2 - 12yz + \cancel{4}z^2 = \cancel{4}z^2$$

$$36x^2+9y^2-12yz = 0$$

$\boxed{12x^2+3y^2-4yz = 0}$ is req. eq. of Cone.

Q16 DTX : $x^2+4y^2-2x+8y-4 = 0$ ; $z = 3$

Vertex $A = (-1,2,1)$

Sol. Let $P(x,y,z)$ be any pt. on req. Cone & let line AP meets dTx at pt. $Q(x_1,y_1,z_1)$

Since pt. Q lies on dTx

so $x_1^2+4y_1^2-2x_1+8y_1-4 = 0$ ; $z_1 = 3$ ——— ①

The eq. of line AQ is

$$\frac{x+1}{x_1+1} = \frac{y-2}{y_1-2} = \frac{z-1}{z_1-1}$$

Put $z_1 = 3$

$$\frac{x+1}{x_1+1} = \frac{y-2}{y_1-2} = \frac{z-1}{3-1}$$

$$\frac{x+1}{x_1+1} = \frac{y-2}{y_1-2} = \frac{z-1}{2}$$

$$\frac{x+1}{x_1+1} = \frac{z-1}{2}, \quad \frac{y-2}{y_1-2} = \frac{z-1}{2}$$

$$\Rightarrow \frac{x_1+1}{x+1} = \frac{2}{z-1}, \quad \frac{y_1-2}{y-2} = \frac{2}{z-1}$$

$$x_1+1 = \frac{2x+2}{z-1}, \quad y_1-2 = \frac{2y-4}{z-1}$$

$$x_1 = \frac{2x+2}{z-1} - 1, \quad y_1 = \frac{2y-4}{z-1} + 2$$

$$x_1 = \frac{2x+2-z+1}{z-1}, \quad y_1 = \frac{2y-4+2z-2}{z-1}$$

$$x_1 = \frac{2x-z+3}{z-1}, \quad y_1 = \frac{2y+2z-6}{z-1}$$

Put in (1)

$$\frac{(2x-z+3)^2}{(z-1)^2} + 4\frac{(2y+2z-6)^2}{(z-1)^2} - 2\cdot\frac{(2x-z+3)}{z-1} + 8\frac{(2y+2z-6)}{(z-1)} - 4 = 0$$

$$(2x-z+3)^2 + 4(2y+2z-6)^2 - 2(2x-z+3)(z-1) + 16(y+z-3)(z-1)$$
$$-4(z-1)^2 = 0$$

$$4x^2+z^2+9-4xz-6z+12x+16(y^2+z^2+9+2yz-6z-6y)$$
$$-2(2xz-z^2+3z-2x+z-3)+16(yz+z^2-3z-y-z+3)$$
$$-4(z^2-2z+1) = 0$$

$4x^2+z^2+9-4xz-6z+12x+16y^2+16z^2+144+32yz$ 179

$-96z-96y-4xz+2z^2-6z+4x-2z+6+16yz+16z^2$

$-48z-16y-16z+48-4z^2+8z-4=0$

$$\boxed{4x^2+16y^2+31z^2-8xz+48yz+16x-112y-166z+203=0}$$

is req. eq. of Cone.

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Q17 Show that the eq. of the Cone with vertex at $(3,1,2)$ & whose dTx $2x^2+3y^2=1$, $z=0$ is $2x^2+3y^2+5z^2-3yz-6zx+z-1=0$

Sol. Let $P(x,y,z)$ be any pt. on req. Cone & let AP meets directrix at pt. $Q(x_1,y_1,z_1)$

Since $Q(x_1,y_1,z_1)$ lies on directrix

So $2x_1^2+3y_1^2=1$ ; $z_1=0$ ———— ①

The eq. of line AQ is

$$\frac{x-3}{x_1-3}=\frac{y-1}{y_1-1}=\frac{z-2}{z_1-2}$$

Put $z_1=0$

$$\frac{x-3}{x_1-3}=\frac{z-2}{-2}, \quad \frac{y-1}{y_1-1}=\frac{z-2}{-2}$$

$$\Rightarrow \frac{x_1-3}{x-3}=-\frac{2}{z-2}, \quad \frac{y_1-1}{y-1}=\frac{-2}{z-2}$$

$$x_1-3=\frac{-2(x-3)}{z-2}, \quad y_1-1=\frac{-2(y-1)}{z-2}$$

$$x_1 = \frac{6-2x}{z-2} + 3 \quad , \quad y_1 = \frac{z-2y}{z-2} + 1$$ 11.

$$x_1 = \frac{\cancel{6}-2x+3z-\cancel{6}}{z-2} \quad , \quad y_1 = \frac{z-2y+z-\cancel{2}}{z-2}$$

$$x_1 = \frac{3z-2x}{z-2} \quad , \quad y_1 = \frac{z-2y}{z-2}$$

using these values in (1)

$$2\left(\frac{3z-2x}{z-2}\right)^2 + 3\left(\frac{z-2y}{z-2}\right)^2 = 1$$

$$2\,\frac{(3z-2x)^2}{(z-2)^2} + 3\,\frac{(z-2y)^2}{(z-2)^2} = 1$$

$$2(3z-2x)^2 + 3(z-2y)^2 = (z-2)^2$$

$$2(9z^2-12zx+4x^2) + 3(z^2-4yz+4y^2) = z^2-4z+4$$

$$18z^2-24zx+8x^2+3z^2-12yz+12y^2-z^2+4z-4=0$$

$$8x^2+12y^2+20z^2-24zx-12yz+4z-4=0$$

$$\boxed{2x^2+3y^2+5z^2-6xz-3yz+z-1=0}$$

is req. eq.

<u>Q18</u> Show that an eq. of the cylinder whose generators intersect the curve 181

$$ax^2+2hxy+by^2+2gx+2fy+c=0, \quad z=0$$

and are parallel to $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is

$$a(nx-lz)^2+2h(nx-lz)(ny-mz)+b(ny-mz)^2$$
$$+2gn(nx-lz)+2nf(ny-mz)+n^2c=0$$

<u>Sol.</u> Let $P(x_1,y_1,z_1)$ be any pt. on the req. cylinder then eq. of line through P & ∥ to $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=t \quad \text{——— (1)}$$

$$\Rightarrow \left.\begin{aligned} x &= x_1+lt \\ y &= y_1+mt \\ z &= z_1+nt \end{aligned}\right]$$

Any pt. on line (1) is

$(x_1+lt, y_1+mt, z_1+nt)$

If this pt. lies on ~~direction~~

then

$$a(x_1+lt)^2+2h(x_1+lt)(y_1+mt)+b(y_1+mt)^2+2g(x_1+lt)$$
$$+2f(y_1+mt)+c=0$$

and $z_1+nt=0$

$$\Rightarrow t=-\frac{z_1}{n}$$

Put in above eqs

$$a\left(x_1-\frac{lz_1}{n}\right)^2+2h\left(x_1-\frac{lz_1}{n}\right)\left(y_1-\frac{mz_1}{n}\right)+b\left(y_1-\frac{mz_1}{n}\right)^2$$
$$+2g\left(x_1-\frac{lz_1}{n}\right)+2f\left(y_1-\frac{mz_1}{n}\right)+c=0$$

$$a\left(\frac{nx_1-lz_1}{n}\right)^2+2h\left(\frac{nx_1-lz_1}{n}\right)\left(\frac{ny_1-mz_1}{n}\right)+b\left(\frac{ny_1-mz_1}{n}\right)^2$$
$$+2g\left(\frac{nx_1-lz_1}{n}\right)+2f\left(\frac{ny_1-mz_1}{n}\right)+c=0$$

$$a(nx_1-lz_1)^2+2h(nx_1-lz_1)(ny_1-mz_1)+b(ny_1-mz_1)^2$$
$$+2gn(nx_1-lz_1)+2nf(ny_1-mz_1)+n^2c=0$$

Now the locus of $(x_1,y_1,z_1)$ is

$$a(nx-lz)^2+2h(nx-lz)(ny-mz)+b(ny-mz)^2$$
$$+2gn(nx-lz)+2nf(ny-mz)+n^2c=0$$

which is eq. of req. cylinder.

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Q19 Show that an eq. of the cylinder whose generators are parallel to $z$-axis & which passes through the curve $x^2+y^2+z^2=1$, $x+y+z=1$ is $x^2+y^2+xy-x-y=0$

Sol: Let $P(x_1,y_1,z_1)$ be any pt. on req. cylinder

Now eq. of line through $(x_1,y_1,z_1)$ & $\parallel$ to $z$-axis is

$$\frac{x-x_1}{0}=\frac{y-y_1}{0}=\frac{z-z_1}{1}=t \quad \text{—— (1)}$$

$$\Rightarrow \left.\begin{aligned} x &= x_1 \\ y &= y_1 \\ z &= z_1+t \end{aligned}\right\}$$

Any pt. on line (1) is $(x_1, y_1, z_1+t)$

If this pt. lies on directrix

$x^2+y^2+z^2=1, \quad x+y+z=1$, so

$$x_1^2+y_1^2+(z_1+t)^2=1 \quad \text{——— (2)}$$

$$x_1+y_1+z_1+t=1 \quad \text{——— (3)}$$

From (3)

$$t = 1-x_1-y_1-z_1$$

Put in (2)

$$x_1^2+y_1^2+(\cancel{z_1}+1-x_1-y_1-\cancel{z_1})^2=1$$

$$x_1^2+y_1^2+1+x_1^2+y_1^2-2x_1+2x_1y_1-2y_1=1$$

$$2x_1^2+2y_1^2+2x_1y_1-2x_1-2y_1=0$$

$$x_1^2+y_1^2+x_1y_1-x_1-y_1=0$$

Now the locus of $(x_1, y_1, z_1)$ is

$$x^2+y^2+xy-x-y=0$$

which is eq. of required cylinder.

Q20 Show that an eq. of the right circular[181] cylinder of radius 3 & whose axis is the line $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$ is

$$5x^2+5y^2+8z^2-8xy+4xz+4yz-6x-42y-96z+225=0$$

Sol:- Here radius of cylinder = 3

The eq. of axis of cylinder is

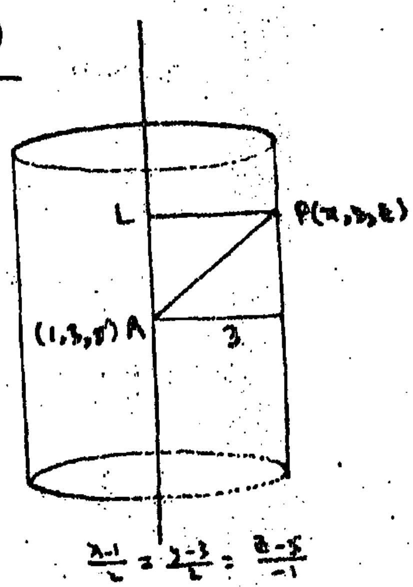
$$\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1} \quad \text{———} \ (1)$$

Its d.rs. are $2, 2, -1$

let $P(x,y,z)$ be any pt. on the surface of cylinder then d.rs. of AP are $x-1, y-3, z-5$ where A is a pt. on axis & let L be the foot of perp. from P on its axis then

$|AL|$ = Projection of AP on axis of cylinder

$$|AL| = \frac{2(x-1)+2(y-3)-1(z-5)}{\sqrt{(2)^2+(2)^2+(-1)^2}}$$




$$= \frac{2x+2y-z-2-6+5}{\sqrt{9}}$$

$$|AL| = \frac{2x+2y-z-3}{3}$$

$$|AL|^2 = \frac{1}{9}(2x+2y-z-3)^2$$

$$|AL|^2 = \frac{1}{9}\Big[(2x+2y)-(z+3)\Big]^2$$

$$= \frac{1}{9}\Big[(2x+2y)^2+(z+3)^2-2(2x+2y)(z+3)\Big]$$

$$= \frac{1}{9}\Big[4x^2+4y^2+8xy+z^2+6z+9-2(2xz+6x+2yz+6y)\Big]$$

$$= \frac{1}{9}\Big[4x^2+4y^2+8xy+z^2+6z+9-4xz-12x-4yz-12y\Big]$$

$$|AL|^2 = \frac{1}{9}\Big[4x^2+4y^2+z^2+8xy-4yz-4xz-12x-12y+6z+9\Big]$$

& $$|AP|^2 = (x-1)^2+(y-3)^2+(z-5)^2$$

$$= x^2-2x+1+y^2-6y+9+z^2-10z+25$$

$$= x^2+y^2+z^2-2x-6y-10z+35$$

& $$|LP|^2 = (3)^2 = 9$$

From right angled $\Delta ALP$

$$|AP|^2 = |AL|^2+|LP|^2$$

$$x^2+y^2+z^2-2x-6y-10z+35 = \frac{1}{9}\Big(4x^2+4y^2+z^2+8xy-4yz-4xz-12x-12y+6z+9\Big)$$

Multiplying both sides by 9

$$9x^2+9y^2+9z^2-18x-54y-90z+315 = 4x^2+4y^2+z^2+8xy-4yz-4xz-12x-12y+6z+9$$

$$\boxed{5x^2+5y^2+8z^2-8xy+4yz+4xz-6x-42y-96z+225=0}$$

is eq. of req. right circular cylinder.

Q21 Show that the eq. to the right circular cone with vertex at O, axis OZ & semi vertical angle $\alpha$ is $x^2+y^2 = z^2\tan^2\alpha$.

Sol.

Let $P(x,y,z)$ be any pt. on the cone then

d.rs. of OP are $x-0, y-0, z-0$,

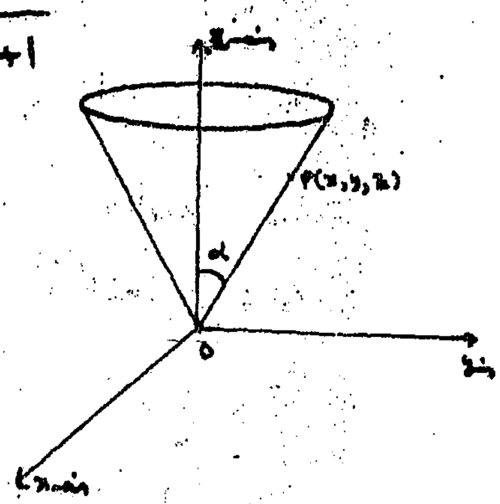
or $x, y, z$

Now d.rs. of OZ-axis (z-axis) are $0, 0, 1$

Here $\alpha$ is the semi vertical angle of cone then

$$\cos\alpha = \frac{x(0)+y(0)+z(1)}{\sqrt{x^2+y^2+z^2}\sqrt{0+0+1}}$$

$$= \frac{0+0+z}{\sqrt{x^2+y^2+z^2}}$$




So $\cos\alpha = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

$$\Rightarrow \cos^2\alpha = \frac{z^2}{x^2+y^2+z^2}$$

$$\frac{1}{\cos^2\alpha} = \frac{x^2+y^2+z^2}{z^2}$$

$$\sec^2\alpha = \frac{x^2+y^2+z^2}{z^2}$$

$$z^2\sec^2\alpha = x^2+y^2+z^2$$

$$z^2(\sec^2\alpha-1) = x^2+y^2$$

$$\boxed{z^2\tan^2\alpha = x^2+y^2}$$

is eq. of req. Cone.

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Q22 Show that the general eq. to the Cone of the second degree which passes through the Co-ord. axes is $fyz+gzx+hxy=0$

Hint

[The general eq. to the Cone of second degree is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ & this is to be satisfied by the d.cs. of the Co-ord. axes i.e., by $1,0,0$ ; $0,1,0$ & $0,0,1$]

Soln

The general eq. to the Cone of second degree is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \text{—(A)}$$

Since this Cone passes through the Co-ord. axes. Therefore the pts. at unit distance from origin on Co-ord. axes lie on Cone (A)

For pt. $(1,0,0)$ Put in (A)

$$\boxed{a = 0}$$

For pt. $(0,1,0)$ Put in (A)

$$\boxed{b=0}$$

For pt. $(0,0,1)$ Put in (A)

$$\boxed{c=0}$$

Using these values of $a, b$ & $c$ in (A)

$$0+0+0+2fyz+2gzx+2hxy = 0$$

or $$2fyz+2gzx+2hxy = 0$$

or $\boxed{fyz+gzx+hxy = 0}$ is eq. of req. Cone.

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Q23 Prove that the eq. to the Cone whose vertex is the origin & which passes through the Curve of intersection of the plane $lx+my+nz=p$ & the surface $ax^2+by^2+cz^2=1$ is $ax^2+by^2+cz^2 = \left(\dfrac{lx+my+nz}{p}\right)^2$

Sol. The eq. of a line through origin is

$$\frac{x-0}{\lambda} = \frac{y-0}{\mu} = \frac{z-0}{\nu}$$

$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu} = t \quad \text{———} (A)$$

$$\Rightarrow \left.\begin{aligned} x &= \lambda t \\ y &= \mu t \\ z &= \nu t \end{aligned}\right]$$

Any pt. on line ① is $(\lambda t, \mu t, \nu t)$

If this pt. lies on the directrix of req. Cone which is

$$lx+my+nz = p, \quad ax^2+by^2+cz^2 = 1$$

then

$$\left.\begin{aligned} l\lambda t + m\mu t + n\nu t &= p \\ \& \ a\lambda^2t^2 + b\mu^2t^2 + c\nu^2t^2 &= 1 \end{aligned}\right]$$

or $t(l\lambda + m\mu + n\nu) = p$ —— ①

& $t^2(a\lambda^2 + b\mu^2 + c\nu^2) = 1$ —— ②

from ① $t = \dfrac{p}{l\lambda + m\mu + n\nu}$

Put in ②

$$\frac{p^2}{(l\lambda+m\mu+n\nu)^2}(a\lambda^2+b\mu^2+c\nu^2) = 1$$

$$p^2(a\lambda^2+b\mu^2+c\nu^2) = (l\lambda+m\mu+n\nu)^2$$

Now eliminating $\lambda, \mu, \nu$ by using ①

$$p^2(ax^2+by^2+cz^2) = (lx+my+nz)^2$$

or $$ax^2+by^2+cz^2 = \left(\frac{lx+my+nz}{p}\right)^2$$

is req. eq. of Cone.

Q.24 The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the Co-ord. axes in A, B & C. Prove that an eq. to the Cone generated by lines drawn from the origin to meet the circle ABC is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

Sol. First of all we will find the eq. of directrix of req. Cone.

For this the Co-ords. of pts. A, B & C are

A(a,0,0), B(0,b,0) & C(0,0,c)

We know the eq. of a sphere through O, A, B & C is $x^2+y^2+z^2-ax-by-cz=0$

So eq. of directrix of Cone is

$$x^2+y^2+z^2-ax-by-cz=0;\quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

The eq. of line through origin is

$$\frac{x-0}{\lambda}=\frac{y-0}{\mu}=\frac{z-0}{\nu}$$

or $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}=t$

$$\Rightarrow \left.\begin{aligned} x&=\lambda t\\ y&=\mu t\\ z&=\nu t\end{aligned}\right\}$$

Any pt. on this line is $(\lambda t, \mu t, \nu t)$

If this pt. lies on direction then 

$$\lambda^2t^2 + \mu^2t^2 + \nu^2t^2 - a\lambda t - b\mu t - c\nu t = 0 \quad \text{——— (1)}$$

$$\frac{\lambda t}{a} + \frac{\mu t}{b} + \frac{\nu t}{c} = 1 \quad \text{——— (2)}$$

from (2) $t\left(\frac{\lambda}{a} + \frac{\mu}{b} + \frac{\nu}{c}\right) = 1$

or $\quad t = \dfrac{1}{\left(\frac{\lambda}{a} + \frac{\mu}{b} + \frac{\nu}{c}\right)}$

Put in (1)

$$\frac{1}{\left(\frac{\lambda}{a}+\frac{\mu}{b}+\frac{\nu}{c}\right)^2}\left(\lambda^2+\mu^2+\nu^2\right) - \frac{1}{\left(\frac{\lambda}{a}+\frac{\mu}{b}+\frac{\nu}{c}\right)}\left(a\lambda+b\mu+c\nu\right) = 0$$

Multiplying both sides by $\left(\frac{\lambda}{a}+\frac{\mu}{b}+\frac{\nu}{c}\right)^2$

$$\lambda^2+\mu^2+\nu^2 - \left(\frac{\lambda}{a}+\frac{\mu}{b}+\frac{\nu}{c}\right)\left(a\lambda+b\mu+c\nu\right) = 0$$

Now eliminating $\lambda, \mu, \nu$ by using eq. (A)

$$x^2+y^2+z^2 - \left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\left(ax+by+cz\right) = 0$$

$$x^2+y^2+z^2 - \left(x^2 + \frac{b}{a}xy + \frac{c}{a}xz + \frac{a}{b}xy + y^2 + \frac{c}{b}yz + \frac{a}{c}xz + \frac{b}{c}yz + z^2\right) = 0$$

$$x^2+y^2+z^2 - \left(x^2+y^2+z^2 + xy\left(\frac{a}{b}+\frac{b}{a}\right) + yz\left(\frac{b}{c}+\frac{c}{b}\right) + xz\left(\frac{a}{c}+\frac{c}{a}\right)\right) = 0$$

$$\cancel{x^2}+\cancel{y^2}+\cancel{z^2}-\cancel{x^2}-\cancel{y^2}-\cancel{z^2} - xy\left(\frac{a}{b}+\frac{b}{a}\right) - yz\left(\frac{b}{c}+\frac{c}{b}\right) - xz\left(\frac{a}{c}+\frac{c}{a}\right) = 0$$

Multiply both sides by $-1$

$$xy\left(\frac{a}{b}+\frac{b}{a}\right) + yz\left(\frac{b}{c}+\frac{c}{b}\right) + xz\left(\frac{a}{c}+\frac{c}{a}\right) = 0$$

is req. eq. of Cone.

Q25 Find an eq. to the cone whose vertex is the origin & directrix, the circle $x=a$, $y^2+z^2=b^2$. Show that the trace of the cone in a plane parallel to xy-plane is a hyperbola.

Sol.
Given eq. of directrix is

$x=a$, $y^2+z^2=b^2$

The eq. of line through origin is

$$\frac{x-0}{\lambda}=\frac{y-0}{\mu}=\frac{z-0}{\nu}$$

$$\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}=t \quad \text{———} \; (A)$$

Any pt. on this line is $(\lambda t, \mu t, \nu t)$

If this pt. lies on dtx then

$$\lambda t = a \quad \text{———} \; (1)$$

$$\mu^2 t^2+\nu^2 t^2 = b^2 \quad \text{———} \; (2)$$

From (1) $t=\frac{a}{\lambda}$

Put in (2)

$$\frac{a^2}{\lambda^2}\mu^2+\frac{a^2}{\lambda^2}\nu^2=b^2$$

$$\frac{a^2}{\lambda^2}(\mu^2+\nu^2)=b^2$$

$$a^2(\mu^2+\nu^2)=b^2\lambda^2$$

Now eliminating $\lambda, \mu, \nu$ by using eq. (A)

$\boxed{a^2(y^2+z^2)=b^2x^2}$ is eq. of req. cone.

For trace of this Cone parallel to xy-plane.

Put $z = k$ in eq. of Cone.

$$a^2(y^2+k^2) = b^2x^2$$

$$a^2y^2 + a^2k^2 = b^2x^2$$

or $$b^2x^2 = a^2y^2 + a^2k^2$$

$$b^2x^2 - a^2y^2 = a^2k^2$$

Dividing both sides by $a^2k^2$

$$\frac{b^2x^2}{a^2k^2} - \frac{a^2y^2}{a^2k^2} = 1$$

or $$\frac{x^2}{\frac{a^2k^2}{b^2}} - \frac{y^2}{k^2} = 1$$

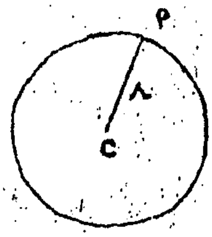
or $$\boxed{\frac{x^2}{\left(\frac{ak}{b}\right)^2} - \frac{y^2}{k^2} = 1}$$ is eq. trace

which is a hyperbola.

**Sphere**: It is the locus of a pt. $P(x,y,z)$ which moves in space such that its distance from a fixed pt. remains constt.

(i) Fixed pt. is called centre of sphere.

(ii) Fixed distance $|CP| = r$ is called radius of sphere.

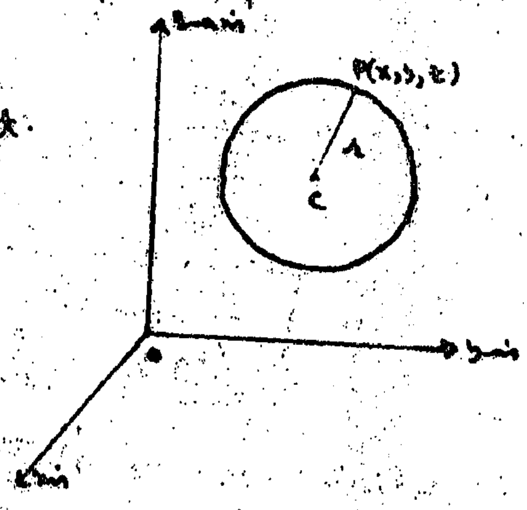



**Eq. of a sphere**

Let $C(a,b,c)$ be the centre of a sphere & $r$ be its radius.

Now take a variable pt. $P(x,y,z)$ on the sphere then by def. of sphere

$$|CP| = r$$




$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2} = r$$

or $\boxed{(x-a)^2+(y-b)^2+(z-c)^2 = r^2}$ is the eq. of a sphere with centre $(a,b,c)$ & radius $r$.

**Note** If centre is at origin then $(a,b,c) = (0,0,0)$

So $x^2+y^2+z^2 = r^2$ is eq. of sphere with centre at origin & radius $r$.

General eq. of a sphere:-

The general eq. of a sphere is

$$x^2+y^2+z^2+2Gx+2Fy+2Kz+C=0 \quad \text{--- (1)}$$

Now we find its centre & radius.

Eq. (1) can be written as

$$(x^2+2Gx)+(y^2+2Fy)+(z^2+2Kz) = -C$$

Adding $G^2+F^2+K^2$ on both sides

$$(x^2+2Gx+G^2)+(y^2+2Fy+F^2)+(z^2+2Kz+K^2) = G^2+F^2+K^2-C$$

$$(x+G)^2+(y+F)^2+(z+K)^2 = \left(\sqrt{G^2+F^2+K^2-C}\right)^2$$

or $\left(x-(-G)\right)^2+\left(y-(-F)\right)^2+\left(z-(-K)\right)^2 = \left(\sqrt{G^2+F^2+K^2-C}\right)^2$

which is the eq. of a sphere.

Its centre is $(-G, -F, -K)$

Its radius is $\sqrt{G^2+F^2+K^2-C}$

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Note An eq. of the form

$$ax^2+by^2+cz^2+2Gx+2Fy+2Kz+C=0$$

will be a sphere if

(i) $a=b=c$

(ii) $a, b$ & $c$ are of same signs.

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