Ex 11·2

Inverse Laplace Transform:

If $F(s)$ is the Laplace transform of a function $f(t)$, then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is denoted by $\mathcal{L}^{-1}\{F(s)\} = f(t)$

Theorem: (Linearity property). If $\mathcal{L}^{-1}\{F_1(s)\} = f_1(t)$ and $\mathcal{L}^{-1}\{F_2(s)\} = f_2(t)$, then

$$\mathcal{L}^{-1}\{aF_1(s) + bF_2(s)\} = a\mathcal{L}^{-1}\{F_1(s)\} + b\mathcal{L}^{-1}\{F_2(s)\}$$

$$= af_1(t) + bf_2(t).$$

Example: Compute $\mathcal{L}^{-1}\left\{\frac{5s}{s^2+5}\right\}$

Solution: Here $\frac{5s}{s^2+5} = 5 \cdot \frac{s}{s^2+(\sqrt{5})^2}$

So from the table of Laplace transforms, we have

$$\mathcal{L}^{-1}\left\{\frac{5s}{s^2+5}\right\} = 5\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{5})^2}\right\}$$

$$= 5\cos\sqrt{5}t.$$

Example: Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-15}\right\}$.

Solution: Here $\frac{1}{s^2+2s-15} = \frac{1}{(s+5)(s-3)}$

$$= \frac{A}{(s+5)} + \frac{B}{s-3}$$

$$= \frac{1}{8}\left[\frac{1}{s-3} - \frac{1}{s+5}\right]$$

$$L^{-1}\left\{\frac{1}{s^2+2s-15}\right\} = \frac{1}{8}L^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{8}L^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= \frac{1}{8}e^{3t} - \frac{1}{8}e^{-5t}.$$

Example: Compute $L^{-1}\left\{\frac{3s+17}{s^2+8s+25}\right\}$

Sol: Here $\frac{3s+17}{s^2+8s+25} = \frac{3(s+4)+5}{(s+4)^2+3^2} = 3\frac{s+4}{(s+4)^2+3^2} + 5\frac{1}{(s+4)^2+3^2}$

So

$$L^{-1}\left\{\frac{3s+17}{s^2+8s+25}\right\} = 3L^{-1}\left\{\frac{s+4}{(s+4)^2+3^2}\right\} + 5L^{-1}\left\{\frac{1}{(s+4)^2+3^2}\right\}$$

$$= 3e^{-4t}L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{5}{3}e^{-4t}L^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= 3e^{-4t}\cos 3t + \frac{5}{3}e^{-4t}\sin 3t.$$

Example: Evaluate $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$

Solution: Here $\frac{s^2}{(s^2+a^2)^2} = \frac{1}{s^2+a^2} - \frac{a^2}{(s^2+a^2)^2}$

So

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - a^2L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$$

$$= \frac{1}{a}\sin at - a^2L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} \quad \text{——(1)}$$

Now

$$L\{t\sin at\} = -\frac{d}{ds}L\{\sin at\}$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{2as}{(s^2+a^2)^2}.$$

So $\mathcal{L}^{-1}\left(\frac{2as}{(s^2+a^2)^2}\right) = t\sin at$

or $\quad 2a\,\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = t\sin at$

Now $\quad 2a\,\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\cdot\frac{1}{s}\right\}$

$$= \int_0^t u\sin au\,du \quad \left(\because \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du\right)$$

$$= -\frac{t\cos at}{a} + \frac{1}{a^2}\sin at$$

Substituting into (1), we have

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \frac{1}{a}\sin at + \frac{t\cos at}{2} - \frac{\sin at}{2a}$$

$$= \frac{1}{2a}(\sin at + at\cos at).$$

Example: Compute $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}$

Solution: Here $\mathcal{L}\{u_2(t)\} = \frac{e^{-2s}}{s}$.

Therefore $\quad \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = \int_0^t u_2(\tau)\,d\tau$

$$= (t-2)\,u_2(t).$$

Again, $\quad \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = \int_0^t (\tau-2)\,u_2(\tau)\,d\tau$

$$= \frac{(t-2)^2}{2}\,u_2(t).$$

Alternatively:

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = u_2(t)f(t-2),$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2} = u_2(t)\cdot\frac{(t-2)^2}{2}$.

<u>Example:</u> Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s^2-a^2)}\right\}$

<u>Solution:</u> We have $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2-a^2)}\right\} = \dfrac{1}{a}\sinh at.$

Now
$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-a^2)}\right\} = \frac{1}{a}\int_0^t \sinh au\,du$$
$$= \frac{1}{a^2}\left[\cosh au\right]_0^t$$
$$= \frac{1}{a^2}\left[\cosh at - 1\right]$$

Applying the same property again, we obtain
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\} = \frac{1}{a^2}\int_0^t(\cosh au - 1)\,du$$
$$= \frac{1}{a^3}\sinh at - \frac{1}{a^2}t.$$

<u>Alternative method:</u>
$$\frac{1}{s^2(s^2-a^2)} = \frac{1}{a^2}\left(\frac{1}{s^2-a^2} - \frac{1}{s^2}\right)$$

Therefore,
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\} = \frac{1}{a^2}\mathcal{L}^{-1}\left\{\frac{1}{(s^2-a^2)}\right\} - \frac{1}{a^2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$
$$= \frac{1}{a^3}\sinh at - \frac{1}{a^2}t$$

<u>Example:</u> compute $\mathcal{L}^{-1}\left\{\dfrac{2a^2}{(s^2+a^2)^2}\right\}$.

<u>Solution:</u> We have $\mathcal{L}^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \sin at.$

By the Convolution property, we get

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$$\mathcal{L}^{-1}\left\{\frac{2a^2}{(s^2+a^2)^2}\right\} = 2(\sin at * \sin at)$$

$$= 2\int_0^t \sin a(t-u)\sin au\, du$$

$$= 2\int_0^t \{\sin at \cos au - \cos at \sin au\} \sin au\, du$$

$$= \sin at \int_0^t 2\cos au \sin au\, du - 2\cos at \int_0^t \sin^2 au\, du$$

$$= \sin at \int_0^t \sin 2au\, du - \cos at \int_0^t 2\sin^2 au\, du$$

$$= -\frac{\sin at\,(\cos 2at - 1)}{2a} - \cos at\left[t - \frac{\sin 2at}{2a}\right]$$

$$= \frac{1}{2a}(\sin 2at \cos at - \cos 2at \sin at) - t\cos at + \frac{1}{2a}\sin at$$

$$= \frac{1}{2a}\sin(2at - at) + \frac{1}{2a}\sin at - t\cos at$$

$$= \frac{1}{a}\sin at - t\cos at.$$

Ex: 11.3 SY

Solution Of Initial Value Problems By the Laplace Transform:

In this section we will use the Laplace transform to solve constant co-efficients linear initial value problems. By the methods of Laplace transform, a differential equation can be converted into an algebraic equation. The independent variable will be t instead of x. If $y(t)$ is a solution of a differential equation then the Laplace transform of $y(t)$ will be denoted by $\mathcal{L}\{y(t)\} = Y(s)$.

The following procedure will be adopted:

(I) Given an initial value problem, take Laplace transform of both sides. Use initial conditions to convert the differential equation into an algebraic equation in $Y(s)$.

(II) Solve the algebraic equation for $Y(s)$.

(III) The inverse transform $\mathcal{L}^{-1}\{Y(s)\} = y(t)$ is the required solution of the given problem.

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(Exercise 11.2)

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Compute the inverse Laplace transform of each of the following (Problems 1–20):

Q1 $\dfrac{s-2}{s^2-2}$

Sol. Let $F(s) = \dfrac{s-2}{s^2-2}$

$$F(s) = \frac{s}{s^2-2} - \frac{2}{s^2-2}$$

Then
$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-2} - \frac{2}{s^2-2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2-(\sqrt{2})^2}\right\} - \mathcal{L}^{-1}\left\{\frac{\sqrt{2}\cdot\sqrt{2}}{s^2-(\sqrt{2})^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2-(\sqrt{2})^2}\right\} - \sqrt{2}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2-(\sqrt{2})^2}\right\}$$

$$= \cosh\sqrt{2}t - \sqrt{2}\sinh\sqrt{2}t$$

Q2 $\dfrac{3s+1}{s^2-6s+18}$

Sol. Let $F(s) = \dfrac{3s+1}{s^2-6s+18}$

$$= \frac{3s+1}{s^2-6s+9+9}$$

$$= \frac{3(s-3)+9+1}{(s-3)^2+9}$$

$$= \frac{3(s-3)+10}{(s-3)^2+(3)^2}$$

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$$= \frac{3(s-3)}{(s-3)^2+(3)^2} + \frac{10}{(s-3)^2+(3)^2}$$

$$F(s) = 3\,\frac{s-3}{(s-3)^2+(3)^2} + \frac{10}{3}\cdot\frac{3}{(s-3)^2+(3)^2}$$

Then $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{ 3\,\frac{s-3}{(s-3)^2+(3)^2} + \frac{10}{3}\,\frac{3}{(s-3)^2+(3)^2} \right\}$

$$= 3\,\mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+(3)^2}\right\} + \frac{10}{3}\,\mathcal{L}^{-1}\left\{\frac{3}{(s-3)^2+(3)^2}\right\}$$

$$= 3e^{3t}\cos 3t + \frac{10}{3}e^{3t}\sin 3t$$

Q3 $\dfrac{9s-67}{s^2-16s+39}$

Sol. Let $F(s) = \dfrac{9s-67}{s^2-16s+64+39-64}$

$$= \frac{9s-67}{(s-8)^2-25}$$

$$= \frac{9(s-8)+72-67}{(s-8)^2-(5)^2}$$

$$= \frac{9(s-8)+5}{(s-8)^2-(5)^2}$$

$$= \frac{9(s-8)}{(s-8)^2-(5)^2} + \frac{5}{(s-8)^2-(5)^2}$$

$$F(s) = 9\,\frac{s-8}{(s-8)^2-(5)^2} + \frac{5}{(s-8)^2-(5)^2}$$

Then $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{ 9\,\frac{s-8}{(s-8)^2-(5)^2} + \frac{5}{(s-2)^2-(5)^2} \right\}$

$$= 5\,\mathcal{L}^{-1}\left\{\frac{s-8}{(s-8)^2-(5)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{(s-8)^2-(5)^2}\right\}$$

$$\mathcal{L}^{-1}\{F(s)\} = 5e^{8t}\cosh 5t + e^{8t}\sinh 5t$$

Q4 $\dfrac{as+b}{s^2+2cs+d}$, $d > c^2 > 0$

Sol.

Let $F(s) = \dfrac{as+b}{s^2+2cs+d}$

$$= \frac{as+b}{s^2+2cs+c^2+d-c^2}$$

$$= \frac{as+b}{(s+c)^2+(d-c^2)} = \frac{as+ac+b-ac}{(s+c)^2+(d-c^2)}$$

$$= \frac{a(s+c)+b-ac}{(s+c)^2+(\sqrt{d-c^2})^2}$$

$$= \frac{a(s+c)}{(s+c)^2+(\sqrt{d-c^2})^2} + \frac{b-ac}{(s+c)^2+(\sqrt{d-c^2})^2}$$

$$F(s) = a\frac{s+c}{(s+c)^2+(\sqrt{d-c^2})^2} + \frac{b-ac}{\sqrt{d-c^2}}\cdot\frac{\sqrt{d-c^2}}{(s+c)^2+(\sqrt{d-c^2})^2}$$

then

$$\mathcal{L}^{-1}\{F(s)\} = a\mathcal{L}^{-1}\left\{\frac{s-(-c)}{(s-(-c))^2+(\sqrt{d-c^2})^2}\right\} + \frac{b-ac}{\sqrt{d-c^2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{d-c^2}}{(s-(-c))^2+(\sqrt{d-c^2})^2}\right\}$$

$$= a e^{-ct}\cos\sqrt{d-c^2}\,t + \frac{b-ac}{\sqrt{d-c^2}}e^{-ct}\sin\sqrt{d-c^2}\,t$$

Q5 $\dfrac{s}{(s+a)^2+b^2}$

Sol. Let $F(s) = \dfrac{s}{(s+a)^2+b^2}$

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$$F(s) = \frac{(s+a)-a}{(s+a)^2+b^2}$$

$$F(s) = \frac{s+a}{(s+a)^2+b^2} - \frac{a}{(s+a)^2+b^2}$$

then

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2} - \frac{a}{b}\frac{b}{(s+a)^2+b^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s-(-a)}{(s-(-a))^2+b^2}\right\} - \frac{a}{b}\mathcal{L}^{-1}\left\{\frac{b}{(s-(-a))^2+b^2}\right\}$$

$$= e^{-at}\cos bt - \frac{a}{b}e^{-at}\sin bt$$

Q6 $\quad \dfrac{1}{(s^2+a^2)(s^2+b^2)}$

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2}$$

$$1 = (As+B)(s^2+a^2) + (Cs+D)(s^2+b^2)$$

$$= As^3 + Aa^2 + Bs^2 + Ba^2 + Cs^3 + Ca^2 + Ds^2 + Da^2$$

Comparing coefficients s^3, s^2, s

$0 = A + C$

$0 = B + D$

Sol. let $F(s) = \dfrac{1}{(s^2+a^2)(s^2+b^2)}$

$$= \frac{1}{(a^2-b^2)}\left[\frac{a^2-b^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{(a^2-b^2)}\left[\frac{(s^2+a^2)-s^2-b^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{(a^2-b^2)}\left[\frac{(s^2+a^2)-(s^2+b^2)}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{(a^2-b^2)}\left[\frac{1}{s^2+b^2} - \frac{1}{s^2+a^2}\right]$$

$$= \frac{1}{b^2-a^2}\left[\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right]$$

$$F(s) = \frac{1}{b^2-a^2}\left[\frac{1}{a}\cdot\frac{a}{s^2+a^2} - \frac{1}{b}\cdot\frac{b}{s^2+b^2}\right]$$

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then $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{b^2-a^2}\left[\frac{1}{a}\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} - \frac{1}{b}\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\}\right]$

$= \frac{1}{b^2-a^2}\left\{\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right\}$

Q7 $\frac{1}{(s-1)(s^2+4)}$

Sol. let $F(s) = \frac{1}{(s-1)(s^2+4)}$ ——— (A)

Consider $\frac{1}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$

or $1 = A(s^2+4)+(Bs+C)(s-1)$ ——— I

To find A put $s=1$ in I

$1 = A(1+4)$

$1 = 5A$

$\Rightarrow \boxed{A = \frac{1}{5}}$

from I

$1 = A(s^2+4) + Bs^2 - Bs + Cs - C$

$1 = (A+B)s^2 + (C-B)s + (4A-C)$

Comparing coeffs. of s^2 & s

$A+B = 0$ ——— (1)

$C-B = 0$ ——— (2)

(1) $\Rightarrow \frac{1}{5} + B = 0 \Rightarrow \boxed{B = -\frac{1}{5}}$

(2) $\Rightarrow C + \frac{1}{5} = 0 \Rightarrow \boxed{C = -\frac{1}{5}}$

S.

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$$\frac{1}{(s-1)(s^2+4)} = \frac{\frac{1}{5}}{s-1} + \frac{-\frac{1}{5}s - \frac{1}{5}}{s^2+4}$$

or
$$\frac{1}{(s-1)(s^2+4)} = \frac{1}{5(s-1)} - \frac{s+1}{5(s^2+4)}$$

So from eq. A

$$F(s) = \frac{1}{5(s-1)} - \frac{s+1}{5(s^2+4)}$$

then

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{1}{s^2+4}\right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+(2)^2}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2+(2)^2}\right\}$$

$$= \frac{1}{5}e^{t} - \frac{1}{5}\cos 2t - \frac{1}{10}\sin 2t$$

Q8 $\dfrac{7s+5}{(3s-8)^2}$

Sol: Let $F(s) = \dfrac{7s+5}{(3s-8)^2}$

$$= \frac{7s+5}{9\left(s-\frac{8}{3}\right)^2} = \frac{7\left[s - \frac{56}{7}\right] + \frac{56}{3} + 5}{9\left(s-\frac{8}{3}\right)^2}$$

$$= \frac{7\left(s-\frac{8}{3}\right) + \frac{56}{3} + 5}{9\left(s-\frac{8}{3}\right)^2}$$

$$= \frac{7\left(s-\frac{8}{3}\right) + \frac{71}{3}}{9\left(s-\frac{8}{3}\right)^2}$$

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$$F(S) = \frac{7}{9(S-\frac{8}{3})} + \frac{71}{27(S-\frac{8}{3})^2}$$

Then $\mathcal{L}^{-1}\{F(S)\} = \frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{S-\frac{8}{3}}\right\} + \frac{71}{27}\mathcal{L}^{-1}\left\{\frac{1}{(S-\frac{8}{3})^2}\right\}$

$$= \frac{7}{9}e^{\frac{8}{3}t} + \frac{71}{27}te^{\frac{8}{3}t}$$

Q9 $\frac{5S+3}{(S+7)^5}$

$$= \frac{5S+3\cdot 5+3-35}{(S+7)^5}$$

Sol. let $F(S) = \frac{5S+3}{(S+7)^5}$

$$= \frac{5(S+7)+3-35}{(S+7)^5}$$

$$= \frac{5(S+7)-32}{(S+7)^5}$$

$$F(S) = \frac{5}{(S+7)^4} - \frac{32}{(S+7)^5}$$

then $\mathcal{L}^{-1}\{F(S)\} = 5\mathcal{L}^{-1}\left\{\frac{1}{(S+7)^4}\right\} - 32\mathcal{L}^{-1}\left\{\frac{1}{(S+7)^5}\right\}$

$$= \frac{5}{6}\mathcal{L}^{-1}\left\{\frac{3!}{(S-(-7))^4}\right\} - \frac{32}{24}\mathcal{L}^{-1}\left\{\frac{4!}{(S-(-7))^5}\right\}$$

$$= \frac{5}{6}t^3e^{-7t} - \frac{4}{3}t^4e^{-7t}$$

$$= t^3e^{-7t}\left(\frac{5}{6} - \frac{4}{3}t\right)$$

Q10 $\frac{2S-3}{2S^3+3S^2-2S-3}$

Sol. let $F(S) = \frac{2S-3}{2S^3+3S^2-2S-3}$ —— (A)

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Consider $\dfrac{2s-3}{2s^3+3s^2-2s-3}$

$$= \frac{2s-3}{2s(s^2-1)+3(s^2-1)}$$

$$= \frac{2s-3}{(s^2-1)(2s+3)}$$

$$= \frac{2s-3}{(s-1)(s+1)(2s+3)}$$

Now

$$\frac{2s-3}{(s-1)(s+1)(2s+3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{2s+3}$$

or $2s-3 = A(s+1)(2s+3) + B(s-1)(2s+3) + C(s-1)(s+1)$ —— I

For A, put $s=1$ in I

$$2-3 = A(2)(5) \Rightarrow \boxed{A = -\frac{1}{10}}$$

For B, put $s=-1$ in I

$$-2-3 = B(-2)(1) \Rightarrow \boxed{B = 5/2}$$

For C, put $s = -3/2$ in I

$$2(-3/2)-3 = C(-3/2-1)(-3/2+1)$$

$$-6 = C(-5/2)(-1/2)$$

$$-6 = C\left(\frac{5}{4}\right) \Rightarrow \boxed{C = -\frac{24}{5}}$$

So $\dfrac{2s-3}{(s-1)(s+1)(2s+3)} = \dfrac{-1}{10(s-1)} + \dfrac{5}{2(s+1)} - \dfrac{24}{5(2s+3)}$

or $\dfrac{2s-3}{2s^3+3s^2-2s-3} = \dfrac{-1}{10(s-1)} + \dfrac{5}{2(s+1)} - \dfrac{24}{5(2s+3)}$

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Put in Ⓐ

$$F(S) = \frac{-1}{10(S-1)} + \frac{5}{2(S+1)} - \frac{24}{5(2S+3)}$$

then

$$\mathcal{L}^{-1}\{F(S)\} = \frac{-1}{10}\mathcal{L}^{-1}\left\{\frac{1}{S-1}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{S-(-1)}\right\} - \frac{24}{5}\mathcal{L}^{-1}\left\{\frac{1}{2(S+3/2)}\right\}$$

$$= \frac{-1}{10}\mathcal{L}^{-1}\left\{\frac{1}{S-1}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{S-(-1)}\right\} - \frac{12}{5}\mathcal{L}^{-1}\left\{\frac{1}{S-(-3/2)}\right\}$$

$$= \frac{-1}{10}e^{t} + \frac{5}{2}e^{-t} - \frac{12}{5}e^{-3/2 t}$$

Q11 $\dfrac{2S^3+6S^2+21S+52}{S(S+2)(S^2+4S+13)}$

Sol.

let $F(S) = \dfrac{2S^3+6S^2+21S+52}{S(S+2)(S^2+4S+13)}$ —— Ⓐ

Consider

$$\frac{2S^3+6S^2+21S+52}{S(S+2)(S^2+4S+13)} = \frac{A}{S} + \frac{B}{S+2} + \frac{CS+D}{S^2+4S+13}$$

$$\Rightarrow 2S^3+6S^2+21S+52 = A(S+2)(S^2+4S+13) + BS(S^2+4S+13) + (CS+D)(S^2+2S)$$

For A put $S=0$

$$52 = A(2)(13) \Rightarrow \boxed{A=2}$$

For B, put $S=-2$

$$2(-8)+6(4)-42+52 = B(-2)(4-8+13)$$

$$-16+24+10 = -2B(9)$$

$$18 = -18B \Rightarrow \boxed{B=-1}$$

From above

$$2S^3+6S^2+21S+52 = A(S^3+4S^2+13S+2S^2+8S+26) + B(S^3+4S^2+13S)$$
$$+ CS^3 + 2CS^2 + DS^2 + 2DS$$

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Equating coeff of s^3 + s^2

$$A + B + C = 2 \quad \text{———} \quad (1)$$

$$6A + 4B + 2C + D = 6 \quad \text{———} \quad (2)$$

$$(1) \Rightarrow \quad 2 - 1 + C = 2$$

$$1 + C = 2$$

$$\boxed{C = 1}$$

$$(2) \Rightarrow \quad 6(2) + 4(-1) + 2(1) + D = 6$$

$$10 + D = 6$$

$$\boxed{D = -4}$$

So

$$\frac{2s^3 + 6s^2 + 21s + 82}{s(s+2)(s^2+4s+13)} = \frac{2}{s} - \frac{1}{s+2} + \frac{s-4}{s^2+4s+13}$$

Put in (A)

$$F(s) = \frac{2}{s} - \frac{1}{s+2} + \frac{s-4}{s^2+4s+13}$$

then

$$\mathcal{L}^{-1}\{F(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{s-4}{s^2+4s+4+9}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{s-4}{(s+2)^2+(3)^2}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\} + \mathcal{L}^{-1}\left\{\frac{(s+2)-6}{(s+2)^2+(3)^2}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\} + \mathcal{L}^{-1}\left\{\frac{s-(-2)}{(s-(-2))^2+(3)^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{3}{(s-(-2))^2+(3)^2}\right\}$$

$$= 2(1) - e^{-2t} + e^{-2t}\cos 3t - 2e^{-2t}\sin 3t$$

1407

Q12 $\dfrac{1}{(s^2+4)(s^2+6s-5)}$

Sol. Let $F(s) = \dfrac{1}{(s^2+4)(s^2+6s-5)}$ ——— (A)

Consider

$$\frac{1}{(s^2+4)(s^2+6s-5)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+6s-5}$$

$$\Rightarrow \quad 1 = (As+B)(s^2+6s-5) + (Cs+D)(s^2+4)$$

$$1 = As^3 + 6As^2 - 5As + Bs^2 + 6Bs - 5B + Cs^3 + 4Cs + Ds^2 + 4D$$

$$1 = (A+C)s^3 + (6A+B+D)s^2 + (-5A+6B+4C)s + (-5B+4D)$$

Comparing coffs. on both sides

$$A + C = 0 \text{ ——— } (1)$$
$$6A + B + D = 0 \text{ ——— } (2)$$
$$-5A + 6B + 4C = 0 \text{ ——— } (3)$$
$$-5B + 4D = 1 \text{ ——— } (4)$$

$4(1) - (3) \Rightarrow 4A + \cancel{4C} + 5A - 6B - \cancel{4C} = 0$

$$9A - 6B = 0$$

or $\quad 3A - 2B = 0$ ——— (5)

$4(2) - (4) \Rightarrow 24A + 4B + \cancel{4D} + 5B - \cancel{4D} = 0 - 1$

$$24A + 9B = -1 \text{ ——— } (6)$$

$8(5) - (6) \Rightarrow \cancel{24A} - 16B - \cancel{24A} - 9B = 0 + 1$

$$-25B = 1 \quad \Rightarrow \quad \boxed{B = \frac{-1}{25}}$$

$(5) \Rightarrow \quad 3A - 2\left(\frac{-1}{25}\right) = 0$

$$3A + \frac{2}{25} = 0$$

$$3A = -\frac{2}{25} \quad \Rightarrow \quad \boxed{A = -\frac{2}{75}}$$

1408

66

① $\Rightarrow$ $-\frac{2}{75}+C=0$ or $\boxed{C = \frac{2}{75}}$

④ $\Rightarrow$ $-5\left(\frac{-1}{25}\right)+4D=1$

$$\frac{1}{5}+4D=1$$

$$4D=1-\frac{1}{5}$$

$$4D=\frac{4}{5} \Rightarrow \boxed{D=\frac{1}{5}}$$

So

$$\frac{1}{(s^2+4)(s^2+6s-5)} = \frac{-\frac{2}{75}s-\frac{1}{25}}{s^2+4} + \frac{\frac{2}{75}s+\frac{1}{5}}{s^2+6s-5}$$

$$= \frac{-2s-3}{75(s^2+4)} + \frac{2s+15}{75(s^2+6s-5)}$$

$$= \frac{-2s}{75(s^2+4)} - \frac{3}{75(s^2+4)} + \frac{2s}{75(s^2+6s-5)} + \frac{15}{75(s^2+6s-5)}$$

$$= -\frac{2}{75}\cdot\frac{s}{s^2+4} - \frac{1}{25}\cdot\frac{1}{s^2+4} + \frac{2}{75}\frac{s}{(s^2+6s-5)} + \frac{1}{5}\cdot\frac{1}{s^2+6s-5}$$

$$= -\frac{2}{75}\cdot\frac{s}{s^2+(2)^2} - \frac{1}{25}\cdot\frac{1}{s^2+(2)^2} + \frac{2}{75}\frac{s}{s^2+6s+9-14} + \frac{1}{5}\cdot\frac{1}{s^2+6s+9-14}$$

$$= -\frac{2}{75}\cdot\frac{s}{s^2+(2)^2} - \frac{1}{25}\cdot\frac{1}{s^2+(2)^2} + \frac{2}{75}\boxed{\frac{(s+3)-3}{(s+3)^2-(\sqrt{14})^2}} + \frac{1}{5}\frac{1}{(s+3)^2-(\sqrt{14})^2}$$

Put in Ⓐ

$$F(s) = -\frac{2}{75}\cdot\frac{s}{s^2+(2)^2} - \frac{1}{25}\cdot\frac{1}{s^2+(2)^2} + \frac{2}{75}\frac{s+3}{(s+3)^2-(\sqrt{14})^2} + \left(-\frac{6}{75}+\frac{1}{5}\right)\frac{1}{(s+3)^2-(\sqrt{14})^2}$$

$$F(s) = -\frac{2}{75}\frac{s}{s^2+(2)^2} - \frac{1}{50}\frac{2}{s^2+(2)^2} + \frac{2}{75}\cdot\frac{s+3}{(s+3)^2-(\sqrt{14})^2} + \frac{3}{25}\frac{1}{(s+3)^2-(\sqrt{14})^2}$$

then

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{2}{75}\mathcal{L}^{-1}\left\{\frac{s}{s^2+(2)^2}\right\} - \frac{1}{50}\mathcal{L}^{-1}\left\{\frac{2}{s^2+(2)^2}\right\} + \frac{2}{75}\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2-(\sqrt{14})^2}\right\} + \frac{3}{25\sqrt{14}}\mathcal{L}^{-1}\left\{\frac{\sqrt{14}}{(s+3)^2-(\sqrt{14})^2}\right\}$$

1409

$$L^{-1}\{F(s)\} = -\frac{2}{75}\cos 2t - \frac{1}{50}\sin 2t + \frac{2}{75}e^{-3t}\cosh(\sqrt{14}\,t) + \frac{3}{75\sqrt{14}}e^{-3t}\sinh\sqrt{14}\,t$$

Q17 $$\frac{s^3+3s^2-s-3}{(s^2+2s+5)^2}$$

Soln. let $F(s) = \dfrac{s^3+3s^2-s-3}{(s^2+2s+5)^2}$ ——— ①

Consider

$$\frac{s^3+3s^2-s-3}{(s^2+2s+5)^2} = \frac{As+B}{s^2+2s+5} + \frac{Cs+D}{(s^2+2s+5)^2}$$

or $s^3+3s^2-s-3 = (As+B)(s^2+2s+5) + (Cs+D)$

$= As^3+2As^2+5As+Bs^2+2Bs+5B+Cs+D$

or $s^3+3s^2-s-3 = As^3+(2A+B)s^2+(5A+2B+C)s+(5B+D)$

Comparing Coeffs. on both sides

$A = 1$ ——— I

$2A+B = 3$ ——— II

$5A+2B+C = -1$ ——— III

$5B+D = -3$ ——— IV

I $\Rightarrow$ $\boxed{A=1}$

II $\Rightarrow$ $2+B=3$ $\Rightarrow$ $\boxed{B=1}$

III $\Rightarrow$ $5+2+C=-1$

or $C=-1-7$ or $\boxed{C=-8}$

IV $\Rightarrow$ $5+D=-3$ or $\boxed{D=-8}$

So $$\frac{s^3+3s^2-s-3}{(s^2+2s+5)^2} = \frac{s+1}{s^2+2s+5} + \frac{-8s-8}{(s^2+2s+5)^2}$$

$$= \frac{s+1}{s^2+2s+5} - \frac{8(s+1)}{(s^2+2s+5)^2}$$

Put in ①

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$$F(s) = \frac{s+1}{s^2+2s+5} - 8\frac{s+1}{(s^2+2s+5)^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\} - 8L^{-1}\left\{\frac{s+1}{(s^2+2s+5)^2}\right\} \quad \text{—— (A)}$$

Now

$$L^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{s+1}{s^2+2s+1+4}\right\}$$

$$= L^{-1}\left\{\frac{s+1}{(s+1)^2+(2)^2}\right\}$$

$$= e^{-t}\cos 2t$$

Consider

$$L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{1}{s^2+2s+1+4}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+1)^2+(2)^2}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{2}{(s+1)^2+(2)^2}\right\}$$

$$L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = \frac{1}{2}e^{-t}\sin 2t$$

Using formula

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}(F(s))\right\}$$

$$L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+2s+5}\right)\right\}$$

$$\frac{1}{2}e^{-t}\sin 2t = -\frac{1}{t}L^{-1}\left\{\frac{-1}{(s^2+2s+5)^2}\cdot(2s+2)\right\}$$

$$\frac{1}{2}t\,e^{-t}\sin 2t = L^{-1}\left\{\frac{2s+2}{(s^2+2s+5)^2}\right\}$$

1411

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+2s+5)^2}\right\} = \frac{1}{4}te^{-t}\sin 2t$$

Putting values in (A)

$$\{(s)\} = e^{-t}\cos 2t - 8\cdot\frac{1}{4}te^{-t}\sin 2t$$

$$= e^{-t}\cos 2t - 2te^{-t}\sin 2t$$

$\boxed{14}$ $\tan^{-1}\left(\frac{a}{s}\right)$

Sol.

Let $F(s) = \tan^{-1}\left(\frac{a}{s}\right)$

using formula

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}(F(s))\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\tan^{-1}\left(\frac{a}{s}\right)\right)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{1+\frac{a^2}{s^2}}\cdot\frac{-a}{s^2}\right\}$$

$$= \cancel{-}\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{\cancel{s^2}}{s^2+a^2}\cdot\frac{\cancel{-}a}{\cancel{s^2}}\right\}$$

$$= \frac{1}{t}\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}$$

$$= \frac{1}{t}\cdot\sin at$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{\sin at}{t}$$

$$\mathcal{L}\{t f(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}$$

$$= -\frac{d}{ds}F(s)$$

$$t f(t) = -\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}$$

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}$$

$$\frac{a}{s} = a s^{-1}$$

$$= a(-1)s^{-1-1}$$

$$= -\frac{a}{s^2}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

14|12

Q15 $\ln \frac{s^2+1}{(s-1)^2}$

Sol. Let $F(s) = \ln \frac{s^2+1}{(s-1)^2}$

Using formula

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\ln \frac{s^2+1}{(s-1)^2}\right)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\ln(s^2+1) - 2\ln(s-1)\right)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} - \frac{2}{s-1}\right\}$$

$$= -\frac{1}{t}\left[2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}\right]$$

$$= -\frac{1}{t}\left[2\cos t - 2e^{t}\right]$$

$$= -\frac{2}{t}\cos t + \frac{2}{t}e^{t}$$

Q16 $\ln \frac{s^2+a^2}{s^2+b^2}$

Sol. Let $F(s) = \ln \frac{s^2+a^2}{s^2+b^2}$

Using formula

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}(F(s))\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\ln \frac{s^2+a^2}{s^2+b^2}\right)\right\}$$

1413

$$-\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left[\ln(s^2+a^2)-\ln(s^2+b^2)\right]\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+a^2}-\frac{2s}{s^2+b^2}\right\}$$

$$= -\frac{1}{t}\left[2\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}-2\mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\}\right]$$

$$= -\frac{1}{t}\left[2\cos at-2\cos bt\right]$$

$$= -\frac{2}{t}\cos at+\frac{2}{t}\cos bt$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{2}{t}(\cos at-\cos bt)$$

Qn $\dfrac{e^{-3s}}{s^2(s^2+9)}$ $\qquad \mathcal{L}^{-1}\{e^{-as}F(s)\} = u_a(t)f(t-a)$

Sol. let $F(s) = \dfrac{e^{-3s}}{s^2(s^2+9)}$

$$= e^{-3s}\left[\frac{1}{s^2(s^2+9)}\right]$$

$$= \frac{e^{-3s}}{9}\left[\frac{1}{s^2}-\frac{1}{s^2+9}\right]$$

or $F(s) = \dfrac{1}{9}\cdot\dfrac{e^{-3s}}{s^2}-\dfrac{1}{9}\dfrac{e^{-3s}}{s^2+9}$

Then $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}-\frac{1}{27}\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{3}{s^2+(3)^2}\right\}$ $\quad \frac{1}{9}u_3(t)\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}_{t\to t-3} - \frac{1}{9}u_3(t)\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}_{t\to}$

$$= \frac{1}{9}u_3(t)\cdot(t-3)-\frac{1}{27}u_3(t)\cdot\sin 3(t-3)$$

$$= \frac{1}{9}u_3(t)\cdot(t-3)-\frac{1}{27}u_3(t)\cdot\sin(3t-9)$$

$$= \frac{1}{9}u_3(t)\,t\Big|_{t\to t-3}-\frac{1}{27}u_3(t)\sin 3t\Big|_{t\to t-3}$$

$$= \frac{1}{9}u_3(t)\,(t-3)-\frac{1}{27}u_3(t)\sin(3t-9)$$

1414

Q19 $e^{-\pi s}\dfrac{s}{s^2-4s+5}$

Sol. Let $F(s) = e^{-\pi s}\dfrac{s}{s^2-4s+5}$

$$= e^{-\pi s}\left[\frac{s}{s^2-4s+4+1}\right]$$

$$= e^{-\pi s}\left[\frac{(s-2)+2}{(s-2)^2+(1)^2}\right]$$

or $F(s) = e^{-\pi s}\left[\dfrac{s-2}{(s-2)^2+(1)^2}+\dfrac{2}{(s-2)^2+(1)^2}\right]$

$$F(s) = e^{-\pi s}\left(\frac{s-2}{(s-2)^2+(1)^2}\right)+2e^{-\pi s}\left(\frac{1}{(s-2)^2+(1)^2}\right)$$

Then

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s-2}{(s-2)^2+(1)^2}\right\}+2\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{(s-2)^2+(1)^2}\right\}$$

$$= U_\pi(t)\cdot e^{2(t-\pi)}\cdot \cos(t-\pi)+2U_\pi(t)\cdot e^{2(t-\pi)}\cdot \sin(t-\pi)$$

$$= U_\pi(t)\cdot e^{2(t-\pi)}\cdot \cos(\pi-t)-2U_\pi(t)\cdot e^{2(t-\pi)}\cdot \sin(\pi-t)$$

$$= -U_\pi(t)\cdot e^{2(t-\pi)}\cdot \cos t-2U_\pi(t)\cdot e^{2(t-\pi)}\cdot \sin t$$

$$= -U_\pi(t)\cdot e^{2(t-\pi)}\left(\cos t+2\sin t\right)$$

Q19 $e^{-2s}\dfrac{s+6}{s^3-5s^2+6s}$

Sol. Let $F(s) = e^{-2s}\dfrac{s+6}{s^3-5s^2+6s}$

$$= e^{-2s}\left[\frac{s+6}{s(s^2-5s+6)}\right]$$

1415

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$$F(s) = e^{-2s}\left[\frac{s+6}{s(s-2)(s-3)}\right] \quad \text{———} \quad (A)$$

Consider

$$\frac{s+6}{s(s-2)(s-3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-3}$$

or $s+6 = A(s-2)(s-3) + Bs(s-3) + Cs(s-2)$

For A, put $s = 0$

$$6 = A(-2)(-3)$$

$$6 = 6A \implies \boxed{A = 1}$$

For B, put $s = 2$

$$2+6 = B(2)(2-3)$$

$$8 = -2B \implies \boxed{B = -4}$$

For C, put $s = 3$

$$3+6 = C(3)(3-2)$$

$$9 = 3C \implies \boxed{C = 3}$$

So

$$\frac{s+6}{s(s-2)(s-3)} = \frac{1}{s} - \frac{4}{s-2} + \frac{3}{s-3}$$

Put in eq. (A)

$$F(s) = e^{-2s}\left[\frac{1}{s} - \frac{4}{s-2} + \frac{3}{s-3}\right]$$

$$F(s) = e^{-2s}\cdot\frac{1}{s} - 4e^{-2s}\cdot\frac{1}{s-2} + 3e^{-2s}\cdot\frac{1}{s-3}$$

Then

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s}\right\} - 4\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s-2}\right\} + 3\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s-3}\right\}$$

$$= u_2(t) - 4u_2(t)\cdot e^{2(t-2)} + 3u_2(t)\cdot e^{3(t-2)}$$

$$= u_2(t)\left(1 - 4e^{2t-4} + 3e^{6t-6}\right)$$

1416

Q20. $e^{-3s} \dfrac{3s-7}{s^2-10s+26}$

$\dfrac{3s-7}{s^2-10s+25-25+26}$

$\dfrac{3s-7}{(s-5)^2+1}$

Sol. Let $F(s) = e^{-3s} \dfrac{3s-7}{s^2-10s+26}$

$= e^{-3s}\left[\dfrac{3s-7}{s^2-10s+25+1}\right]$

$= e^{-3s}\left[\dfrac{3(s-5)+15-7}{(s-5)^2+(1)^2}\right]$

$= e^{-3s}\left[\dfrac{3(s-5)+8}{(s-5)^2+(1)^2}\right]$

$= e^{-3s}\left[\dfrac{3(s-5)}{(s-5)^2+(1)^2} + \dfrac{8}{(s-5)^2+(1)^2}\right]$

$$F(s) = 3e^{-3s}\frac{s-5}{(s-5)^2+(1)^2} + 8e^{-3s}\frac{1}{(s-5)^2+(1)^2}$$

then

$$\mathcal{L}^{-1}\{F(s)\} = 3\mathcal{L}^{-1}\left\{e^{-3s}\frac{s-5}{(s-5)^2+(1)^2}\right\} + 8\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{(s-5)^2+(1)^2}\right\}$$

$$= 3u_3(t)\, e^{5(t-3)} \cos(t-3) + 8u_3(t)\, e^{5(t-3)} \sin(t-3)$$

In each of problems 21—23, use the Convolution property to evaluate the inverse Laplace transform:

Q21 $\dfrac{1}{s^2(s+5)}$

Sol. Let $F(s) = \dfrac{1}{s^2}$ & $G(s) = \dfrac{1}{s+5}$

then $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$

& $g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{\frac{1}{s+5}\} = e^{-5t}$

1417

By: Convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = f(t) * g(t)$$

$$= \int_0^t u\, e^{-5(t-u)}\, du$$

$$= \int_0^t u \cdot e^{-5t+5u}\, du$$

$$= \int_0^t e^{-5t} \cdot u \cdot e^{5u}\, du$$

$$= e^{-5t}\int_0^t u \cdot e^{5u}\, du$$

$$= e^{-5t}\left[\left| u \cdot \frac{e^{5u}}{5}\right|_0^t - \int_0^t \frac{e^{5u}}{5} \cdot 1\, du\right]$$

$$= e^{-5t}\left[t \cdot \frac{e^{5t}}{5} - \frac{1}{5}\left|\frac{e^{5u}}{5}\right|_0^t\right]$$

$$= e^{-5t}\left[\frac{t e^{5t}}{5} - \frac{1}{25}\left(e^{5t} - e^{0}\right)\right]$$

$$= e^{-5t}\left[\frac{t \cdot e^{5t}}{5} - \frac{1}{25}\left(e^{5t} - 1\right)\right]$$

$$= e^{-5t}\left[\frac{t e^{5t}}{5} - \frac{1}{25}e^{5t} + \frac{1}{25}\right]$$

$$= \frac{t}{5} - \frac{1}{25} + \frac{1}{25}e^{-5t}$$

$$= \frac{5t - 1 + e^{-5t}}{25}$$

$$= \frac{1}{25}\left(e^{-5t} + 5t - 1\right)$$

2nd

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t e^{-5u}(t-u)\, du$$

$$= \int_0^t \cancel{e^{-5u}(t) - e^{-5u}}\, du$$

$$= \int_0^t e^{-5u}\, t$$

1418

76

Imp Q22 $\dfrac{s}{(s+1)(s^2+4)}$

Sol.

Let $F(s) = \dfrac{1}{s+1}$ & $G(s) = \dfrac{s}{s^2+4}$

then $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\} = e^{-t}$

& $g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+(2)^2}\right\} = \cos 2t$

By Convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\} = f(t) * g(t)$$

$$= g(t) * f(t)$$

$$= \int_0^t \cos 2u \cdot e^{-(t-u)}\, du$$

$$= \int_0^t \cos 2u \cdot e^{-t+u}\, du$$

$$= e^{-t}\int_0^t \cos 2u \cdot e^{u}\, du \quad \text{——— (1)}$$

Consider $e^{-t}\int_0^t \cos 2u \cdot e^{u}\, du$

Integ. by parts.

$$= e^{-t}\left[\left|\cos 2u \cdot e^{u}\right|_0^t - \int_0^t e^{u} \cdot -2\sin 2u\, du\right]$$

$$= e^{-t}\left[(\cos 2t \cdot e^{t} - 1) + 2\int_0^t \sin 2u \cdot e^{u}\, du\right]$$

$$= e^{-t}(\cos 2t \cdot e^{t} - 1) + 2e^{-t}\int_0^t \sin 2u \cdot e^{u}\, du$$

$$\text{s.}\ e^{-t}\int_0^t \cos 2u \cdot e^{u}\, du = e^{-t}(\cos 2t \cdot e^{t} - 1) + 2e^{-t}\left[\left|\sin 2u\, e^{u}\right|_0^t - \int_0^t e^{u} \cdot 2\cos 2u\, du\right]$$

$$e^{-t}\int_0^t \cos 2u \cdot e^{u}\,du = e^{-t}\left(e^{t}\cos 2t - 1\right) + e^{-t}\left(\sin 2t \cdot e^{t}\right) - 4e^{-t}\int_0^t \cos 2u \cdot e^{u}\,du$$

$$5e^{-t}\int_0^t \cos 2u \cdot e^{u}\,du = \cos 2t - e^{-t} + 2\sin 2t$$

$$e^{-t}\int_0^t \cos 2u \cdot e^{u}\,du = \frac{1}{5}\left[\cos 2t + 2\sin 2t - e^{-t}\right]$$

Put in 1

$$L^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\} = \frac{1}{5}\left[\cos 2t + 2\sin 2t - e^{-t}\right]$$

Imp

Q23 $\dfrac{1}{(s^2+1)(s^2+4s+5)}$

Sol.

let $F(s) = \dfrac{1}{s^2+1}$

& $G(s) = \dfrac{1}{s^2+4s+5}$

$= \dfrac{1}{s^2+4s+4+1}$

or $G(s) = \dfrac{1}{(s+2)^2+(1)^2}$

then $f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\dfrac{1}{s^2+(1)^2}\right\} = \sin t$

& $g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\dfrac{1}{(s+2)^2+(1)^2}\right\} = e^{-2t}\cdot\sin t$

By the Convolution theorem

$$L^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} = f(t) * g(t)$$

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$$= \int_0^t \sin u \cdot e^{-2(t-u)} \cdot \sin(t-u)\, du$$

$$= \int_0^t \sin u \cdot e^{-2t+2u} \cdot \sin(t-u)\, du$$

$-2\sin\alpha \sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$

$$= \int_0^t e^{-2t} \cdot e^{2u} \cdot (\sin u \sin(t-u))\, du$$

$$= -\frac{e^{-2t}}{2}\int_0^t e^{2u}\cdot(-2\sin u \cdot \sin(t-u))\, du$$

$$= -\frac{e^{-2t}}{2}\int_0^t e^{2u}\left(\cos(u+t-u) - \cos(u-t+u)\right) du$$

$$= -\frac{e^{-2t}}{2}\int_0^t e^{2u}\left(\cos t - \cos(2u-t)\right) du$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+8)}\right\} = -\frac{e^{-2t}}{2}\int_0^t e^{2u}\cos t\, du + \frac{e^{-2t}}{2}\int_0^t e^{2u}\cos(2u-t)\, dt$$

$$= -\frac{e^{-2t}\cos t}{2}\int_0^t e^{2u}\, du - \frac{e^{-2t}}{2}\int_0^t e^{2u}\cdot\cos(2u-t)\, dt$$

$$= -\frac{e^{-2t}\cos t}{2}\left|\frac{e^{2u}}{2}\right|_0^t - \frac{e^{-2t}}{2}\left|\frac{e^{2u}}{4+4}\left(2\cos(2u-t) + 2\sin(2u-t)\right)\right|_0^t$$

$$\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos(bx+c) + b\sin(bx+c)\right)$$

$$= -\frac{e^{-2t}\cos t}{4}\left(e^{2t} - e^{0}\right) - \frac{e^{-2t}}{16}\left[e^{2t}\cdot(2\cos t + 2\sin t) - e^{0}\left(2\cos(-t) + 2\sin(-t)\right)\right]$$

$$= -\frac{e^{-2t}\cos t}{4}\left(e^{2t} - 1\right) - \frac{e^{-2t}}{16}\left[2e^{2t}(\cos t + \sin t) - (2\cos t + 2\sin t)\right]$$

$$= -\frac{\cos t}{4} + \frac{e^{-2t}\cos t}{4} - \frac{1}{8}(\cos t + \sin t) + \frac{e^{-2t}}{8}(\cos t + \sin t)$$ — Ans

Imp

Qn Show that

$$\mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \cosh at \cdot \cos at$$

Sol: Consider

$$\mathcal{L}\{\cosh at \cdot \cos at\} = \mathcal{L}\left\{\left(\frac{e^{at}+e^{-at}}{2}\right)\cos at\right\}$$

$$= \mathcal{L}\left\{\frac{1}{2}(e^{at}\cos at) + \frac{1}{2}(e^{-at}\cos at)\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\cos at\} + \frac{1}{2}\mathcal{L}\{e^{-at}\cos at\}$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2}\right] + \frac{1}{2}\left[\frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)[(s+a)^2+a^2]+(s+a)[(s-a)^2+a^2]}{[(s-a)^2+a^2][(s+a)^2+a^2]}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)[s^2+2as+2a^2]+(s+a)[s^2-2as+2a^2]}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{1}{2}\left[\frac{s^3+\cancel{2as^2}+\cancel{2a^2s}-\cancel{as^2}-\cancel{2a^2s}-\cancel{2a^3}+s^3-\cancel{2as^2}+\cancel{2a^2s}+\cancel{as^2}-\cancel{2a^2s}+\cancel{2a^3}}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$$

$$= \frac{1}{2}\left[\frac{2s^3}{(s^2+2a^2)^2-4a^2s^2}\right]$$

$$= \frac{s^3}{s^4+\cancel{4a^2s^2}+4a^4-\cancel{4a^2s^2}}$$

$$\mathcal{L}\{\cosh at \cos at\} = \frac{s^3}{s^4+4a^4}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \cosh at \cos at$$

Q25 Show that $\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{2a^2}\sinh at \sin at$

Sol: Consider

$$\mathcal{L}\{\sinh at \sin at\} = \mathcal{L}\left\{\left(\frac{e^{at}-e^{-at}}{2}\right)\sin at\right\}$$

$$= \mathcal{L}\left\{\frac{1}{2}e^{at}\sin at - \frac{1}{2}e^{-at}\sin at\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\sin at\} - \frac{1}{2}\mathcal{L}\{e^{-at}\sin at\}$$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2}\right] - \frac{1}{2}\left[\frac{a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right]$$

$\frac{a}{s^2+as}$

$$= \frac{1}{2}a\left[\frac{[(s+a)^2+a^2]-[(s-a)^2+a^2]}{[(s-a)^2+a^2][(s+a)^2+a^2]}\right]$$

$$= \frac{1}{2}a\left[\frac{(s^2+2as+2a^2)-(s^2-2as+2a^2)}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{1}{2}a\left[\frac{\cancel{s^2}+2as+\cancel{2a^2}-\cancel{s^2}+2as-\cancel{2a^2}}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$$

$$= \frac{1}{2}a\left[\frac{4as}{(s^2+2a^2)^2-(2as)^2}\right]$$

$$= \frac{2a^2 s}{s^4+\cancel{4a^2s^2}+4a^4-\cancel{4a^2s^2}}$$

$$\mathcal{L}\{\sinh at \sin at\} = \frac{2a^2 s}{s^4+4a^4}$$

$$\Rightarrow \sinh at \sin at = \mathcal{L}^{-1}\left\{\frac{2a^2 s}{s^4+4a^4}\right\}$$

or $\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{2a^2}\sinh at \cdot \sin at$