Lecture 30: Discrete Mathematics

Course Title: Discrete Mathematics **Course Code:** MTH211 **Class:** BSM-II

Objectives

The main aim of the lecture is to define the notion of

- *combinations with examples.*
- *pigeonhole principle and generalized pigeonhole principle with example.*
- *The inclusion–exclusion pprinciple with example.*

References:

- S. Lipschutz and M. Lipson, Schaum's Outlines Discrete Mathematics, Third Edition, McGraw-Hil, 2007.
- K.H. Rosen, Discrete Mathematics and its Application, MeGraw-Hill, 6th edition. 2007.
- K.A. Ross, C.R.B. Wright, Discrete Mathematics, Prentice Hall. New Jersey, 2003.
- https://www.freepik.com/ (for background image)

Combinations

Let S be a set with n elements. A *combination* of these n elements taken r at a time is any selection of r of the elements where order does not count. Such a selection is called an *r-combination*; it is simply a subset of S with r elements.

The number of such combinations will be denoted by $C(n, r)$.

Other texts may use ${}_nC_r$, $C_{n,r}$, C_r^n or $\binom{n}{r}$.

Before we give the general formula for $C(n, r)$, we consider a special case.

Example: Find the number of combinations of 4 objects, A, B, C, D, taken 3 at a time.
Each combination of three objects determines 3! = 6 permutations of the objects as follows:

$$ABC: ABC, ACB, BAC, BCA, CAB, CBA$$
$$ABD: ABD, ADB, BAD, BDA, DAB, DBA$$
$$ACD: ACD, ADC, CAD, CDA, DAC, DCA$$
$$BCD: BDC, BDC, CBD, CDB, DBC, DCB$$

Thus the number of combinations multiplied by 3! gives us the number of permutations; that is,

$$C(4, 3) \cdot 3! = P(4, 3) \text{ or } C(4, 3) = \frac{P(4,3)}{3!}.$$

But $P(4, 3) = 4 \cdot 3 \cdot 2 = 24$ and $3! = 6$; hence $C(4, 3) = 4$ as noted above.

As indicated above, any combination of n objects taken r at a time determines $r!$ permutations of the objects in the combination; that is,

$$P(n, r) = r!\, C(n, r).$$

Accordingly, we obtain the following formula for $C(n, r)$ which we formally state as a theorem.

Theorem: For $n \geq r > 0$**,** $C(n,r) = \dfrac{P(n,r)}{r!} = \dfrac{n!}{r!(n-r)!}$**.**

Example: A farmer buys 3 cows, 2 sheep, and 4 hens from a man who has 6 cows, 5 sheep, and 8 hens. Find the number *m* of choices that the farmer has.

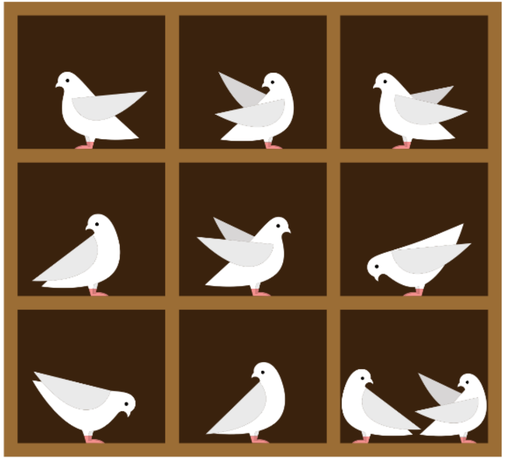
The farmer can choose the cows in *C*(6, 3) ways, the sheep in *C*(5, 2) ways, and the hens in *C*(8, 4) ways. Thus the number *m* of choices follows:

$$m = C(6,3) \cdot C(5,2) \cdot C(8,4) = 20 \cdot 10 \cdot 70 = 14000.$$

The Pigeonhole Principle

Many results in combinational theory come from the following almost obvious statement.

Pigeonhole Principle: If *n* pigeonholes are occupied by *n* + 1 or more pigeons, then at least one pigeonhole is occupied by more than one pigeon.



This principle can be applied to many problems where we want to show that a given situation can occur.

Example

Suppose a department contains 13 professors, then two of the professors (pigeons) were born in the same month (pigeonholes).

The Pigeonhole Principle is generalized as follows.

Generalized Pigeonhole Principle: If n pigeonholes are occupied by $kn + 1$ or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by $k + 1$ or more pigeons.

Example:

Find the minimum number of students in a class to be sure that three of them are born in the same month.

Solution: Here the $n = 12$ months are the pigeonholes, and $k + 1 = 3$ so $k = 2$.

Hence among any $kn + 1 = 25$ students (pigeons), three of them are born in the same month.

The Inclusion–Exclusion Principle

Let A and B be any finite sets. Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

In other words, to find the number $n(A \cup B)$ of elements in the union of A and B, we add $n(A)$ and $n(B)$ and then we subtract $n(A \cap B)$; that is, we "include" $n(A)$ and $n(B)$, and we "exclude" $n(A \cap B)$. This follows from the simple fact that, when we add $n(A)$ and $n(B)$, we have counted the elements of set $(A \cap B)$ twice when A and B are not disjoint sets.

The above principle holds for any number of sets. We first state it for three sets.

Theorem: For any finite sets A, B, C we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

That is, we "include" $n(A)$, $n(B)$, $n(C)$, we "exclude" $n(A \cap B)$, $n(A \cap C)$, $n(B \cap C)$, and finally "include" $n(A \cap B \cap C)$.

Example

Find the number of mathematics students at a college taking at least one of the languages French, German, and Russian, given the following data:

65 study French, 20 study French and German,

45 study German, 25 study French and Russian, 8 study all three languages.

42 study Russian, 15 study German and Russian,

Solution: We want to find $n(F \cup G \cup R)$, where F, G, and R denote the sets of students studying French, German, and Russian, respectively.

By the Inclusion–Exclusion Principle,

$$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$$

$$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$$

Namely, 100 students study at least one of the three languages.

THANKS FOR YOUR ATTENTION