Ch 01: Improper Integrals of 1st and 2nd Kinds

Course Title: Real Analysis II | Course Code: MTH322
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We discussed (in MTH321: Real Analysis I) Riemann-Stieltjes's integrals of the form $\int_a^b f\,d\alpha$ under the restrictions that both f and α are defined and bounded on a finite interval $[a,b]$. The integral of the form $\int_a^b f\,d\alpha$ are called definite integrals. To extend the concept, we shall relax some condition on definite integral like f on finite interval or boundedness of f on finite interval.

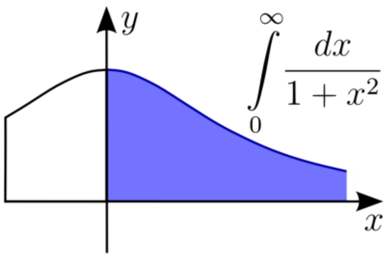
➢ *Definition*

The integral $\int_a^b f\,d\alpha$ is called an improper integral of first kind if $a=-\infty$ or $b=+\infty$ or *both* i.e. one or both integration limits is infinite.

➢ *Definition*

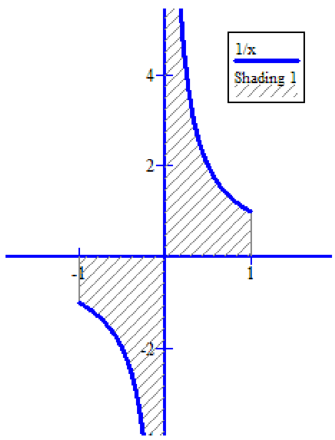
The integral $\int_a^b f\,d\alpha$ is called an improper integral of second kind if $f(x)$ is unbounded at one or more points of $a\le x\le b$. Such points are called singularities of $f(x)$.

➢ *Examples*



- $\int_0^\infty \frac{1}{1+x^2}dx$, $\int_{-\infty}^1 \frac{1}{x-2}dx$ and $\int_{-\infty}^\infty (x^2+1)dx$ are examples of improper integrals of first kind.

- $\int_{-1}^1 \frac{1}{x}dx$ and $\int_0^1 \frac{1}{2x-1}dx$ are examples of improper integrals of second kind.



➢ *Notations*

We shall denote the set of all functions f such that $f\in R(\alpha)$ on $[a,b]$ by $R(\alpha;a,b)$. When $\alpha(x)=x$, we shall simply write $R(a,b)$ for this set. The notation $\alpha\uparrow$ on $[a,\infty)$ will mean that α is monotonically increasing on $[a,\infty)$.

IMPROPER INTEGRAL OF THE FIRST KIND

➢ Definition

Assume that $f \in R(\alpha;a,b)$ for every $b \geq a$. Keep a,α and f fixed and define a function I on $[a,\infty)$ as follows:

$$I(b) = \int_a^b f(x)\,d\alpha(x) \quad \text{if} \quad b \geq a \;\ldots\ldots\ldots\ldots\; (i)$$

The function I so defined is called an infinite (or an improper) integral of first kind and is denoted by the symbol $\int_a^\infty f(x)\,d\alpha(x)$ or by $\int_a^\infty f\,d\alpha$.

The integral $\int_a^\infty f\,d\alpha$ is said to converge if the limit

$$\lim_{b\to\infty} I(b) \;\ldots\ldots\ldots\ldots\; (ii)$$

exists (finite). Otherwise, $\int_a^\infty f\,d\alpha$ is said to diverge.

If the limit in (ii) exists and equals A, the number A is called the value of the integral and we write $\int_a^\infty f\,d\alpha = A$

➢ Example

Consider and integral $\int_1^\infty x^{-p}\,dx$, where p is any real number. Discuss its convergence and divergence.

Solution

Let $I(b) = \int_1^b x^{-p}\,dx$ where $b \geq 1$.

Then $I(b) = \int_1^b x^{-p}\,dx = \left.\dfrac{x^{1-p}}{1-p}\right|_1^b = \dfrac{1-b^{1-p}}{p-1}$ if $p \neq 1$.

If $b \to \infty$, then $b^{1-p} \to 0$ for $p > 1$ and $b^{1-p} \to \infty$ for $p < 1$. Therefore we have

$$\lim_{b\to\infty} I(b) = \lim_{b\to\infty} \frac{1-b^{1-p}}{p-1} = \begin{cases} \infty & \text{if} \quad p < 1, \\ \dfrac{1}{p-1} & \text{if} \quad p > 1. \end{cases}$$

Now if $p = 1$, we get $\int_1^b x^{-1}\,dx = \log b \to \infty$ as $b \to \infty$.

Hence we concluded: $\displaystyle\int_1^\infty x^{-p}\,dx = \begin{cases} \text{diverges} & \text{if} \quad p \leq 1, \\ \dfrac{1}{p-1} & \text{if} \quad p > 1. \end{cases}$

➢ *Example*

Is the integral $\int_0^\infty \sin 2\pi x\,dx$ converges or diverges?

Solution:

Consider $I(b)=\int_0^b \sin 2\pi x\,dx$, where $b\geq 0$.

We have $\int_0^b \sin 2\pi x\,dx = \left.\frac{-\cos 2\pi x}{2\pi}\right|_0^b = \frac{1-\cos 2\pi b}{2\pi}$.

Also $\cos 2\pi b \to l$ as $b\to\infty$, where l has values between -1 and 1, that is, limit is not unique.

Therefore the integral $\int_0^\infty \sin 2\pi x\,dx$ diverges.

➢ *Note*

If $\int_{-\infty}^{a} f\,d\alpha$ and $\int_{a}^{\infty} f\,d\alpha$ are both convergent for some value of a, we say that the integral $\int_{-\infty}^{\infty} f\,d\alpha$ is convergent and its value is defined to be the sum

$$\int_{-\infty}^{\infty} f\,d\alpha = \int_{-\infty}^{a} f\,d\alpha + \int_{a}^{\infty} f\,d\alpha$$

The choice of the point a is clearly immaterial.

If the integral $\int_{-\infty}^{\infty} f\,d\alpha$ converges, its value is equal to the limit: $\lim\limits_{b\to+\infty}\int_{-b}^{b} f\,d\alpha$.

➢ *Review:*

- A function f is said to be increasing, if for all $x_1, x_2 \in D_f$ (domain of f) and $x_1 \leq x_2$ implies $f(x_1)\leq f(x_2)$.
- A function f is said to be bounded if there exist some positive number μ such that $|f(t)|\leq \mu$ for all $t\in D_f$.
- If $\lim\limits_{x\to\infty} f(x)$ exists then f is bounded.
- If $f\in R(\alpha; a,b)$ and $c\in[a,b]$, then $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$.
- If $f\in R(\alpha; a,b)$ and $f(x)\geq 0$ for all $x\in[a,b]$, then $\int_a^b f d\alpha \geq 0$.
- If f is monotonically increasing and bounded on $[a,+\infty)$, then $\lim\limits_{x\to\infty} f(x) = \sup\limits_{x\in[a,\infty)} f(x)$.

> ### ➢ *Theorem*
> Assume that α is monotonically increasing on $[a,+\infty)$ and suppose that $f \in R(\alpha;a,b)$ for every $b \ge a$. Assume that $f(x) \ge 0$ for each $x \ge a$. Then $\int_a^{\infty} f\,d\alpha$ converges if, and only if, there exists a constant $M > 0$ such that
> $$\int_a^b f\,d\alpha \le M \quad \text{for every } b \ge a.$$

Proof

Let $I(b) = \int_a^b f\,d\alpha$ for $b \ge a$.

First suppose that $\int_a^{\infty} f\,d\alpha$ is convergent, then $\lim\limits_{b\to+\infty} I(b)$ exists, that is, $I(b)$ is bounded.

So there exists a constant $M > 0$ such that

$$|I(b)| < M \text{ for every } b \ge a.$$

As $f(x) \ge 0$ for each $x \ge a$, therefore $\int_a^b f\,d\alpha \ge 0$.

This gives $I(b) = \int_a^b f\,d\alpha \le M$ for every $b \ge a$.

Conversely, suppose that there exists a constant $M > 0$ such that $\int_a^b f\,d\alpha \le M$ for every $b \ge a$. This give $|I(b)| \le M$ for every $b \ge a$.

That is, I is bounded on $[a,+\infty)$.

Now for $b_2 \ge b_1 > a$, we have

$$I(b_2) = \int_a^{b_2} f\,d\alpha = \int_a^{b_1} f\,d\alpha + \int_{b_1}^{b_2} f\,d\alpha$$

$$\ge \int_a^{b_1} f\,d\alpha = I(b_1) \qquad \because \int_{b_1}^{b_2} f\,d\alpha \ge 0 \text{ as } f(x) \ge 0 \text{ for all } x \ge a.$$

This gives I is monotonically increasing on $[a,+\infty)$.

As I is monotonically increasing and bounded on $[a,+\infty)$, therefore it is convergent, that is $\int_a^{\infty} f\,d\alpha$ converges.

> **Theorem: (Comparison Test)**
>
> Assume that α is monotonically increasing on $[a,+\infty)$ and $f \in R(\alpha;a,b)$ for every $b \geq a$. If $0 \leq f(x) \leq g(x)$ for every $x \geq a$ and $\int_a^\infty g\, d\alpha$ converges, then $\int_a^\infty f\, d\alpha$ converges and we have
>
> $$\int_a^\infty f\, d\alpha \leq \int_a^\infty g\, d\alpha$$

Proof

Let $I_1(b) = \int_a^b f\, d\alpha$ and $I_2(b) = \int_a^b g\, d\alpha$, $b \geq a$.

$$\because \quad 0 \leq f(x) \leq g(x) \quad \text{for every} \quad x \geq a$$

$$\therefore \quad I_1(b) \leq I_2(b) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$\because \int_a^\infty g\, d\alpha \text{ converges} \quad \therefore \ \exists \text{ a constant } M > 0 \text{ such that}$$

$$\int_a^b g\, d\alpha \leq M \quad , \quad b \geq a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

From (i) and (ii) we have $I_1(b) \leq M$ for every $b \geq a$.

$\Rightarrow \lim\limits_{b\to\infty} I_1(b)$ exists and is finite.

$\Rightarrow \int_a^\infty f\, d\alpha$ converges.

Also $\lim\limits_{b\to\infty} I_1(b) \leq \lim\limits_{b\to\infty} I_2(b) \leq M$

$$\Rightarrow \int_a^\infty f\, d\alpha \leq \int_a^\infty g\, d\alpha .$$

> ***Example***

Is the improper integral $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ convergent or divergent?

Solution:

Since $\sin^2 x \leq 1$ for all $x \in [1,+\infty)$, therefore $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ for all $x \in [1,+\infty)$.

This gives $\int_1^\infty \frac{\sin^2 x}{x^2} dx \leq \int_1^\infty \frac{1}{x^2} dx$.

Now $\int_1^\infty \frac{1}{x^2} dx$ is convergent, therefore $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ is convergent.

> ***Theorem (Limit Comparison Test)***
>
> Assume that α is monotonically increasing on $[a,+\infty)$. Suppose that $f \in R(\alpha;a,b)$ and that $g \in R(\alpha;a,b)$ for every $b \geq a$, where $f(x) \geq 0$ and $g(x) \geq 0$ for $x \geq a$. If
>
> $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$$
>
> then $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ both converge or both diverge.

Proof

For all $b \geq a$, we can find some $N > 0$ such that

$$\left|\frac{f(x)}{g(x)} - 1\right| < \varepsilon \qquad \forall \ x \geq N \quad \text{for every} \quad \varepsilon > 0.$$

$$\Rightarrow\ 1 - \varepsilon < \frac{f(x)}{g(x)} < 1 + \varepsilon$$

Let $\varepsilon = \dfrac{1}{2}$. Then we have

$$\frac{1}{2} < \frac{f(x)}{g(x)} < \frac{3}{2}.$$

$\Rightarrow\ g(x) < 2f(x)$ ……….(i) and $2f(x) < 3g(x)$ ……….(ii)

From (i) $\int_a^\infty g\,d\alpha < 2\int_a^\infty f\,d\alpha$,

$\Rightarrow\ \int_a^\infty g\,d\alpha$ converges if $\int_a^\infty f\,d\alpha$ converges and $\int_a^\infty g\,d\alpha$ diverges if $\int_a^\infty f\,d\alpha$ diverges.

From (ii) $2\int_a^\infty f\,d\alpha < 3\int_a^\infty g\,d\alpha$,

$\Rightarrow\ \int_a^\infty f\,d\alpha$ converges if $\int_a^\infty g\,d\alpha$ converges and $\int_a^\infty g\,d\alpha$ diverges if $\int_a^\infty f\,d\alpha$ diverges.

$\Rightarrow$ The integrals $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ converge or diverge together.

> ***Note***

The above theorem also holds if $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = c$, provided that $c \neq 0$. If $c = 0$, we can only conclude that convergence of $\int_a^\infty g\,d\alpha$ implies convergence of $\int_a^\infty f\,d\alpha$.

➢ *Example*

For every real p, the integral $\int_1^\infty e^{-x}x^p\,dx$ converges.

This can be seen by comparison of this integral with $\int_1^\infty \frac{1}{x^2}\,dx$.

Let $f(x) = e^{-x}x^p$ and $g(x) = \frac{1}{x^2}$.

Now $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = \lim\limits_{x\to\infty}\frac{e^{-x}x^p}{1/x^2}$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty} e^{-x}x^{p+2} = \lim_{x\to\infty}\frac{x^{p+2}}{e^x} = 0. \quad \text{(find this limit yourself)}$$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent, therefore the given integral $\int_1^\infty e^{-x}x^p\,dx$ is also convergent.

➢ *Remark*

It is easy to show that if $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ are convergent, then

- $\int_a^\infty (f \pm g)\,d\alpha$ is convergent.
- $\int_a^\infty cf\,d\alpha$, where c is some constant, is convergent.

➢ *Note*

An improper integral $\int_a^\infty f\,d\alpha$ is said to converge absolutely if $\int_a^\infty |f|\,d\alpha$ converges.

It is said to be convergent conditionally if $\int_a^\infty f\,d\alpha$ converges but $\int_a^\infty |f|\,d\alpha$ diverges.

➢ *Theorem*

Assume $\alpha \uparrow$ on $[a,+\infty)$. If $f \in R(\alpha;a,b)$ for every $b \geq a$ and if $\int_a^\infty |f|\,d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ also converges.

Or: An absolutely convergent integral is convergent.

Proof

If $x \geq a$, $\pm f(x) \leq \left| f(x) \right|$

$\Rightarrow \left| f(x) \right| - f(x) \geq 0 \qquad \Rightarrow 0 \leq \left| f(x) \right| - f(x) \leq 2\left| f(x) \right|$

$\Rightarrow \int_a^\infty \left(\left| f \right| - f \right) d\alpha$ converges.

Now difference of $\int_a^\infty \left| f \right| d\alpha$ and $\int_a^\infty \left(\left| f \right| - f \right) d\alpha$ is convergent,

that is, $\int_a^\infty f\, d\alpha$ is convergent.

➢ *Remark*

Every absolutely convergent integral is convergent.

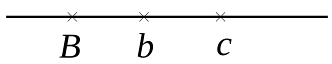
➢ *Theorem (Cauchy condition for infinite integrals)*

Assume that $f \in R(\alpha; a, b)$ for every $b \geq a$. Then the integral $\int_a^\infty f\, d\alpha$ converges if, and only if, for every $\varepsilon > 0$ there exists a $B > 0$ such that $c > b > B$ implies

$$\left| \int_b^c f\, d\alpha \right| < \varepsilon$$

Proof

Let $\int_a^\infty f\, d\alpha$ be convergent, that is $\lim_{b \to \infty} \int_a^b f\, d\alpha = \int_a^\infty f\, d\alpha$.



Then $\exists$ $B > 0$ such that

$$\left| \int_a^b f\, d\alpha - \int_a^\infty f\, d\alpha \right| < \frac{\varepsilon}{2} \quad \text{for every } b \geq B \quad \ldots\ldots\ldots.(i)$$

Also for $c > b > B$,

$$\left| \int_a^c f\, d\alpha - \int_a^\infty f\, d\alpha \right| < \frac{\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ldots.\ (ii)$$

As we know $\int_a^c f\, d\alpha = \int_a^b f\, d\alpha + \int_b^c f\, d\alpha$, this gives

$$\left| \int_b^c f\, d\alpha \right| = \left| \int_a^c f\, d\alpha - \int_a^b f\, d\alpha \right|$$

$$= \left| \int_a^c f\, d\alpha - \int_a^\infty f\, d\alpha + \int_a^\infty f\, d\alpha - \int_a^b f\, d\alpha \right|$$

$$\le \left| \int_a^c f\,d\alpha - \int_a^\infty f\,d\alpha \right| + \left| \int_a^\infty f\,d\alpha - \int_a^b f\,d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \left| \int_b^c f\,d\alpha \right| < \varepsilon \quad \text{when} \quad c > b > B.$$

Conversely, assume that the Cauchy condition holds.

Define $a_n = \int_a^{a+n} f\,d\alpha$ if $n = 1, 2, \ldots\ldots$

Consider n, m such that $a+n, a+m > b > B$, then

$$|a_n - a_m| = \left| \int_a^{a+n} f\,d\alpha - \int_a^{a+m} f\,d\alpha \right| = \left| \int_a^b f\,d\alpha + \int_b^{a+n} f\,d\alpha - \int_a^b f\,d\alpha - \int_b^{a+m} f\,d\alpha \right|$$

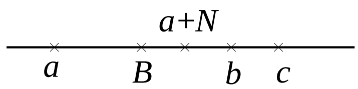
$$= \left| \int_b^{a+n} f\,d\alpha - \int_b^{a+m} f\,d\alpha \right| \le \left| \int_b^{a+n} f\,d\alpha \right| + \left| \int_b^{a+m} f\,d\alpha \right| < \varepsilon + \varepsilon = 2\varepsilon$$

This gives, the sequence $\{a_n\}$ is a Cauchy sequence $\Rightarrow$ it is convergent.

Let $\lim_{n\to\infty} a_n = A$

Given $\varepsilon > 0$, choose B so that $\left| \int_b^c f\,d\alpha \right| < \frac{\varepsilon}{2}$ if $c > b > B$.

and also that $|a_n - A| < \frac{\varepsilon}{2}$ whenever $a + n \ge B$.



Choose an integer N such that $a + N > B$.
Then, if $b > a + N$, we have

$$\left| \int_a^b f\,d\alpha - A \right| = \left| \int_a^{a+N} f\,d\alpha - A + \int_{a+N}^b f\,d\alpha \right|$$

$$\le |a_N - A| + \left| \int_{a+N}^b f\,d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \int_a^\infty f\,d\alpha = A$$

This completes the proof.

➢ *Remarks*

It follows from the above theorem that convergence of $\int_a^\infty f\,d\alpha$ implies $\lim_{b\to\infty} \int_b^{b+\varepsilon} f\,d\alpha = 0$ for every fixed $\varepsilon > 0$.

However, this does not imply that $f(x) \to 0$ as $x \to \infty$.

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IMPROPER INTEGRAL OF THE SECOND KIND

➢ ***Definition***

Let f be defined on the half open interval $(a,b]$ and assume that $f \in R(\alpha; x,b)$ for every $x \in (a,b]$. Define a function I on $(a,b]$ as follows:

$$I(x) = \int_x^b f\, d\alpha \quad \text{if} \quad x \in (a,b] \text{ ……….. } (i)$$

The function I so defined is called an improper integral of the second kind and is denoted by the symbol $\int_{a+}^{b} f(t)\, d\alpha(t)$ or $\int_{a+}^{b} f\, d\alpha$.

The integral $\int_{a+}^{b} f\, d\alpha$ is said to converge if the limit

$$\lim_{x\to a+} I(x) \text{ ……..}(ii) \quad \text{exists (finite).}$$

Otherwise, $\int_{a+}^{b} f\, d\alpha$ is said to diverge. If the limit in (ii) exists and equals A, the number A is called the value of the integral and we write $\int_{a+}^{b} f\, d\alpha = A$.

Similarly, if f is defined on $[a,b)$ and $f \in R(\alpha; a,x)\ \forall\ x \in [a,b)$ then $I(x) = \int_a^x f\, d\alpha$ if $x \in [a,b)$ is also an improper integral of the second kind and is denoted as $\int_a^{b-} f\, d\alpha$ and is convergent if $\lim_{x\to b-} I(x)$ exists (finite).

➢ ***Note***

The tests developed to check the behaviour of the improper integrals of Ist kind are applicable to improper integrals of IInd kind after making necessary modifications.

➢ ***Example***

$f(x) = x^{-p}$ is defined on $(0,b]$ and $f \in R(x,b)$ for every $x \in (0,b]$.

$$I(x) = \int_x^b x^{-p}\, dx \qquad \text{if} \quad x \in (0,b]$$

$$= \int_{0+}^{b} x^{-p}\, dx = \lim_{\varepsilon\to 0} \int_{0+\varepsilon}^{b} x^{-p}\, dx$$

$$= \lim_{\varepsilon\to 0} \left| \frac{x^{1-p}}{1-p} \right|_{\varepsilon}^{b} = \lim_{\varepsilon\to 0} \frac{b^{1-p} - \varepsilon^{1-p}}{1-p} \qquad , \qquad (p \neq 1)$$

$$= \begin{cases} finite, & p<1 \\ infinite, & p>1 \end{cases}$$

When $p=1$, we get $\int_{\varepsilon}^{b} \frac{1}{x}\, dx = \log b - \log\varepsilon \to \infty$ as $\varepsilon \to 0$.

$\Rightarrow \int_{0+}^{b} x^{-1}\, dx$ also diverges.

Hence the integral converges when $p<1$ and diverges when $p \geq 1$.

➢ *Note*

If the two integrals $\int_{a+}^{c} f\, d\alpha$ and $\int_{c}^{b-} f\, d\alpha$ both converge, we write

$$\int_{a+}^{b-} f\, d\alpha = \int_{a+}^{c} f\, d\alpha + \int_{c}^{b-} f\, d\alpha$$

The definition can be extended to cover the case of any finite number of sums. We can also consider mixed combinations such as

$$\int_{a+}^{b} f\, d\alpha + \int_{b}^{\infty} f\, d\alpha \quad \text{which can be written as} \quad \int_{a+}^{\infty} f\, d\alpha.$$

➢ *Assignment*

Consider $\Gamma(p) = \int_{0}^{\infty} e^{-x} x^{p-1}\, dx$, $(p>0)$. Evaluate the convergence of this improper integral.

➢ *A Useful Comparison Integral*

$$\int_{a}^{b} \frac{dx}{(x-a)^n}$$

We have, if $n \neq 1$,

$$\int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^n} = \left| \frac{1}{(1-n)(x-a)^{n-1}} \right|_{a+\varepsilon}^{b}$$

$$= \frac{1}{(1-n)} \left(\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}} \right)$$

Which tends to $\frac{1}{(1-n)(b-a)^{n-1}}$ or $+\infty$ according as $n<1$ or $n>1$, as $\varepsilon \to 0$.

Again, if $n=1$,

$$\int_{a+\varepsilon}^{b} \frac{dx}{x-a} = \log(b-a) - \log\varepsilon \to +\infty \quad \text{as} \quad \varepsilon \to 0.$$

Hence the improper integral $\int_a^b \frac{dx}{(x-a)^n}$ converges iff $n<1$.

➢ *Question*

Examine the convergence of

(*i*) $\int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$ (*ii*) $\int_0^1 \frac{dx}{x^2(1+x)^2}$ (*iii*) $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$

Solution

(*i*) $\int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$

Here '0' is the only point of infinite discontinuity of the integrand.
We have

$$f(x)=\frac{1}{x^{1/3}\left(1+x^2\right)}$$

Take $g(x)=\frac{1}{x^{1/3}}$

Then $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{1}{1+x^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ have identical behaviours.

$\because \int_0^1 \frac{dx}{x^{1/3}}$ converges $\therefore \int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$ also converges.

(*ii*) $\int_0^1 \frac{dx}{x^2(1+x)^2}$

Here '0' is the only point of infinite discontinuity of the given integrand.
We have

$$f(x)=\frac{1}{x^2(1+x)^2}$$

Take $g(x)=\frac{1}{x^2}$

Then $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{1}{(1+x)^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ behave alike.

But $n=2$ being greater than 1, the integral $\int_0^1 g(x)\,dx$ does not converge. Hence the given integral also does not converge.

(*iii*) $\displaystyle\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$

Here '0' and '1' are the two points of infinite discontinuity of the integrand.
We have

$$f(x)=\frac{1}{x^{1/2}(1-x)^{1/3}}$$

We take any number between 0 and 1, say $1/2$, and examine the convergence of the improper integrals $\displaystyle\int_0^{1/2} f(x)\,dx$ and $\displaystyle\int_{1/2}^{1} f(x)\,dx$.

To examine the convergence of $\displaystyle\int_0^{1/2} \frac{1}{x^{1/2}(1-x)^{1/3}}dx$, we take $g(x)=\dfrac{1}{x^{1/2}}$

Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{(1-x)^{1/3}}=1$$

$\because$ $\displaystyle\int_0^{1/2}\frac{1}{x^{1/2}}dx$ converges $\therefore$ $\displaystyle\int_0^{1/2}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

To examine the convergence of $\displaystyle\int_{1/2}^{1} \frac{1}{x^{1/2}(1-x)^{1/3}}dx$, we take $g(x)=\dfrac{1}{(1-x)^{1/3}}$

Then

$$\lim_{x\to 1}\frac{f(x)}{g(x)}=\lim_{x\to 1}\frac{1}{x^{1/2}}=1$$

$\because$ $\displaystyle\int_{1/2}^{1}\frac{1}{(1-x)^{1/3}}dx$ converges $\because$ $\displaystyle\int_{1/2}^{1}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

Hence $\int_0^1 f(x)\,dx$ converges.

➢ *Question*

Show that $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}dx$ exists iff m, n are both positive.

Solution

The integral is proper if $m\ge 1$ and $n\ge 1$.

The number '0' is a point of infinite discontinuity if $m<1$ and the number '1' is a point of infinite discontinuity if $n<1$.

Let $m<1$ and $n<1$.

We take any number, say $1/2$, between 0 & 1 and examine the convergence of the improper integrals $\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ and $\int_{1/2}^{1} x^{m-1}(1-x)^{n-1}dx$ at '0' and '1' respectively.

Convergence at 0:

We write

$$f(x)=x^{m-1}(1-x)^{n-1}=\frac{(1-x)^{n-1}}{x^{1-m}} \quad \text{and take} \quad g(x)=\frac{1}{x^{1-m}}$$

Then $\dfrac{f(x)}{g(x)}\to 1$ as $x\to 0$

As $\int_0^{1/2}\frac{1}{x^{1-m}}\,dx$ is convergent at 0 iff $1-m<1$ i.e. $m>0$

We deduce that the integral $\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ is convergent at 0, iff m is +ive.

Convergence at 1:

We write $f(x)=x^{m-1}(1-x)^{n-1}=\dfrac{x^{m-1}}{(1-x)^{1-n}}$ and take $g(x)=\dfrac{1}{(1-x)^{1-n}}$

Then $\dfrac{f(x)}{g(x)}\to 1$ as $x\to 1$

As $\int_{1/2}^{1}\frac{1}{(1-x)^{1-n}}dx$ is convergent, iff $1-n<1$ i.e. $n>0$.

We deduce that the integral $\int_{1/2}^{1} x^{m-1}(1-x)^{n-1}dx$ converges iff $n>0$.

Thus $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ exists for positive values of m, n only.

It is a function which depends upon m & n and is defined for all positive values of m & n. It is called Beta function.

➢ ***Question***

Show that the following improper integrals are convergent.

(i) $\int_1^\infty \sin^2\frac{1}{x}\,dx$ (ii) $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ (iii) $\int_0^1 \frac{x\log x}{(1+x)^2}\,dx$ (iv) $\int_0^1 \log x\cdot\log(1+x)\,dx$

Solution

(*i*) Let $f(x)=\sin^2\dfrac{1}{x}$ and $g(x)=\dfrac{1}{x^2}$

then $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\dfrac{\sin^2\frac{1}{x}}{\frac{1}{x^2}}=\lim\limits_{y\to 0}\left(\dfrac{\sin y}{y}\right)^2=1$

$\Rightarrow \int_1^\infty f(x)\,dx$ and $\int_1^\infty \dfrac{1}{x^2}\,dx$ behave alike.

$\because \int_1^\infty \dfrac{1}{x^2}\,dx$ is convergent $\therefore \int_1^\infty \sin^2\dfrac{1}{x}\,dx$ is also convergent.

(*ii*) $\int_1^\infty \dfrac{\sin^2 x}{x^2}\,dx$

Take $f(x)=\dfrac{\sin^2 x}{x^2}$ and $g(x)=\dfrac{1}{x^2}$

$\sin^2 x\le 1 \quad\Rightarrow \dfrac{\sin^2 x}{x^2}\le\dfrac{1}{x^2} \quad \forall\ x\in(1,\infty)$

and $\int_1^\infty \dfrac{1}{x^2}\,dx$ converges $\therefore \int_1^\infty \dfrac{\sin^2 x}{x^2}\,dx$ converges.

➢ *Note*

$\int_0^1 \dfrac{\sin^2 x}{x^2}\,dx$ is a proper integral because $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{x^2}=1$ so that '0' is not a point of infinite discontinuity. Therefore $\int_0^\infty \dfrac{\sin^2 x}{x^2}\,dx$ is convergent.

(*iii*) $\int_0^1 \dfrac{x\log x}{(1+x)^2}\,dx$

$\because\ \log x < x\ ,\quad x\in(0,1)$

$\therefore\ x\log x < x^2$

$\Rightarrow\ \dfrac{x\log x}{(1+x)^2} < \dfrac{x^2}{(1+x)^2}$

Now $\int_0^1 \dfrac{x^2}{(1+x)^2}\,dx$ is a proper integral.

$\therefore \int_0^1 \dfrac{x\log x}{(1+x)^2}\,dx$ is convergent.

(*iv*) $\int_0^1 \log x\cdot\log(1+x)\,dx$

$\because\ \log x < x \quad \therefore\ \log(x+1) < x+1$

$\Rightarrow\ \log x \cdot \log(1+x) < x(x+1)$

$\because\ \int_0^1 x(x+1)\,dx$ is a proper integral $\quad \therefore\ \int_0^1 \log x \cdot \log(1+x)\,dx$ is convergent.

➢ *Note*

(*i*) $\int_0^a \frac{1}{x^p}\,dx$ diverges when $p \geq 1$ and converges when $p < 1$.

(*ii*) $\int_a^\infty \frac{1}{x^p}\,dx$ converges iff $p > 1$.

➢ *Questions*

Examine the convergence of

(*i*) $\int_1^\infty \frac{x}{(1+x)^3}\,dx \qquad$ (*ii*) $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}\,dx \qquad$ (*iii*) $\int_1^\infty \frac{dx}{x^{1/3}(1+x)^{1/2}}$

Solution

(***i***) Let $f(x) = \frac{x}{(1+x)^3}$ and take $g(x) = \frac{1}{x^2}$.

As $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^3}{(1+x)^3} = 1$

Therefore the two integrals $\int_1^\infty \frac{x}{(1+x)^3}\,dx$ and $\int_1^\infty \frac{1}{x^2}\,dx$ have identical behaviour for convergence at ∞.

$\because\ \int_1^\infty \frac{1}{x^2}\,dx$ is convergent $\quad \therefore\ \int_1^\infty \frac{x}{(1+x)^3}\,dx$ is convergent.

(***ii***) Let $f(x) = \frac{1}{(1+x)\sqrt{x}}$ and take $g(x) = \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}$

We have $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{1+x} = 1$

and $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ is convergent. Thus $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}\,dx$ is convergent.

(***iii***) Let $f(x) = \frac{1}{x^{1/3}(1+x)^{1/2}}$

we take $g(x) = \frac{1}{x^{1/3} \cdot x^{1/2}} = \frac{1}{x^{5/6}}$

We have $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=1$ and $\int\limits_1^\infty \dfrac{1}{x^{5/6}}dx$ is convergent $\therefore$ $\int\limits_1^\infty f(x)\,dx$ is convergent.

➢ ***Question***

Show that $\int\limits_{-\infty}^{\infty}\dfrac{1}{1+x^2}dx$ is convergent.

Solution

We have

$$\int\limits_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\lim_{a\to\infty}\left[\int\limits_{-a}^{0}\frac{1}{1+x^2}dx+\int\limits_{0}^{a}\frac{1}{1+x^2}dx\right]$$

$$=\lim_{a\to\infty}\left[\int\limits_{0}^{a}\frac{1}{1+x^2}dx+\int\limits_{0}^{a}\frac{1}{1+x^2}dx\right]=2\lim_{a\to\infty}\left[\int\limits_{0}^{a}\frac{1}{1+x^2}dx\right]$$

$$=2\lim_{a\to\infty}\left|\tan^{-1}x\right|_0^a=2\left(\frac{\pi}{2}\right)=\pi$$

therefore the integral is convergent.

➢ ***Question***

Show that $\int\limits_0^\infty\dfrac{\tan^{-1}x}{1+x^2}dx$ is convergent.

Solution

$\because$ $(1+x^2)\cdot\dfrac{\tan^{-1}x}{(1+x^2)}=\tan^{-1}x\to\dfrac{\pi}{2}$ as $x\to\infty$

$\int\limits_0^\infty\dfrac{\tan^{-1}x}{1+x^2}dx$ & $\int\limits_0^\infty\dfrac{1}{1+x^2}dx$ behave alike.

Here $f(x)=\dfrac{\tan^{-1}x}{1+x^2}$

and $g(x)=\dfrac{1}{1+x^2}$

$\because$ $\int\limits_0^\infty\dfrac{1}{1+x^2}dx$ is convergent $\therefore$ A given integral is convergent.

➢ ***Question***

Show that $\int\limits_0^\infty e^{-x}\cos x\,dx$ is absolutely convergent.

Solution

$\because$ $\left|e^{-x}\cos x\right|<e^{-x}$ and $\int\limits_0^\infty e^{-x}\,dx=1$

$\therefore$ the given integral is absolutely convergent. (Comparison test)

➢ *Question*

Show that $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{1-x^4}}\,dx$ is convergent.

Solution

$\because\ e^{-x} < 1$ and $1+x^2 > 1$ for all $x \in (0,1)$.

$$\therefore\ \frac{e^{-x}}{\sqrt{1-x^4}} < \frac{1}{\sqrt{(1-x^2)(1+x^2)}} < \frac{1}{\sqrt{1-x^2}}$$

Also $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx$

$$= \lim_{\varepsilon\to 0} \sin^{-1}(1-\varepsilon) = \frac{\pi}{2}$$

$\Rightarrow \displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{1-x^4}}\,dx$ is convergent. (by comparison test)

References:

(1) Book
Mathematical Analysis
Tom M. Apostol (John Wiley & Sons, Inc.)