DEPARTMENT OF MATHEMATICS
COMSATS University Islamabad, Attock Campus

Trigonometric Functions
Sample Questions: Set 02

1. Only sketch the proof that there exist functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that
   (i) $C''(x) = -C(x)$ and $S''(x) = -S(x)$ for all $x \in \mathbb{R}$
   (ii) $C(0) = 1$, $C'(0) = 0$ and $S(0) = 0$, $S'(0) = 1$.

2. Consider a sequence of functions $\{C_n(x)\}$ define by
$$C_{n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x \in \mathbb{R}.$$
   Prove that $\{C_n\}$ converges uniformly on the interval $[-A, A]$, where $A > 0$.

3. Consider a sequence of functions $\{S_n(x)\}$ define by
$$S_{n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x \in \mathbb{R}.$$
   Prove that $\{S_n\}$ converges uniformly on the interval $[-A, A]$, where $A > 0$.

4. Consider the functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that
   (i) $C''(x) = -C(x)$ and $S''(x) = -S(x)$ for all $x \in \mathbb{R}$
   (ii) $C(0) = 1$, $C'(0) = 0$ and $S(0) = 0$, $S'(0) = 1$.
   Prove that $C^2(x) + S^2(x) = 1$ for $x \in \mathbb{R}$.

5. Consider the functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that
   (i) $C''(x) = -C(x)$ and $S''(x) = -S(x)$ for all $x \in \mathbb{R}$
   (ii) $C(0) = 1$, $C'(0) = 0$ and $S(0) = 0$, $S'(0) = 1$.
   Prove that the functions C and S defined in this way are unique.

6. Define cosine and sine functions.

7. If $f : \mathbb{R} \to \mathbb{R}$ is such that
$$f''(x) = -f(x) \quad \text{for} \quad x \in \mathbb{R},$$
   then there exist real numbers α and β such that
$$f(x) = \alpha C(x) + \beta S(x) \quad \text{for} \quad x \in \mathbb{R},$$
   where C and S represents cosine and sine function.
   Above question can be rephrases as follows:

8. If $f : \mathbb{R} \to \mathbb{R}$ is such that
$$f''(x) = -f(x) \quad \text{for} \quad x \in \mathbb{R},$$
   then there exist real numbers α and β such that
$$f(x) = \alpha \cos x + \beta \sin x \quad \text{for} \quad x \in \mathbb{R},$$

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