Chapter 6 – Riemann-Stieltjes Integral

Course Title: Real Analysis 1 — Course Code: MTH321

Course instructor: Dr. Atiq ur Rehman — Class: MSc-II

➢ Partition

Let $[a,b]$ be a given interval. A finite set $P=\{a=x_0,x_1,x_2,\ldots,x_k,\ldots,x_n=b\}$ is said to be a partition of $[a,b]$ which divides it into n such intervals

$$[x_0,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},x_n].$$

Each sub-interval is called a *component* of the partition.

Obviously, corresponding to different choices of the points x_i we shall have different partition.

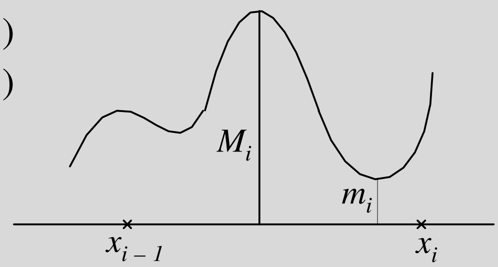
The maximum of the length of the components is defined as the *norm* of the partition.

➢ Riemann Integral

Let f be a real-valued function defined and bounded on $[a,b]$. Corresponding to each partition P of $[a,b]$, we put

$$M_i=\sup f(x) \qquad (x_{i-1}\le x\le x_i)$$

$$m_i=\inf f(x) \qquad (x_{i-1}\le x\le x_i)$$



We define upper and lower sums as

$$U(P,f)=\sum_{i=1}^{n}M_i\,\Delta x_i$$

and $$L(P,f)=\sum_{i=1}^{n}m_i\,\Delta x_i\,.$$

where $\Delta x_i=x_i-x_{i-1} \quad (i=1,2,\ldots,n)$

and finally $$\overline{\int_a^b} f\,dx=\inf U(P,f) \;\ldots\ldots\ldots\ldots\ldots\; (i)$$

$$\underline{\int_a^b} f\,dx=\sup L(P,f) \;\ldots\ldots\ldots\ldots\ldots\;(ii)$$

Where the infimum and the supremum are taken over all partitions P of $[a,b]$.

Then $\overline{\int_a^b} f\,dx$ and $\underline{\int_a^b} f\,dx$ are called the upper and lower Riemann Integrals of f over $[a,b]$ respectively.

In case the upper and lower integrals are equal, we say that f is Riemann-Integrable on $[a,b]$ and we write $f\in R$, where R denotes the set of Riemann integrable functions.

The common value of (i) and (ii) is denoted by $\int_a^b f\,dx$ or by $\int_a^b f(x)\,dx$.

Which is known as the Riemann integral of f over $[a,b]$.

> **➢ *Theorem***
> The upper and lower integrals are defined for every bounded function f.

Proof

Take M and m to be the upper and lower bounds of $f(x)$ in $[a,b]$.

$$\Rightarrow \quad m \le f(x) \le M \qquad (a \le x \le b)$$

Then $M_i \le M$ and $m_i \ge m$ $\qquad (i = 1,2,.....,n)$

Where M_i and m_i denote the supremum and infimum of $f(x)$ in (x_{i-1}, x_i) for certain partition P of $[a,b]$.

$$\Rightarrow \; L(P,f) = \sum_{i=1}^{n} m_i \,\Delta x_i \ge \sum_{i=1}^{n} m\,\Delta x_i \qquad (\Delta x_i = x_{i-1} - x_i)$$

$$\Rightarrow \; L(P,f) \ge m \sum_{i=1}^{n} \Delta x_i$$

But $\displaystyle\sum_{i=1}^{n} \Delta x_i = (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + ... + (x_n - x_{n-1})$

$$= x_n - x_0 = b - a$$

$$\Rightarrow \; L(P,f) \ge m(b-a)$$

Similarity $U(P,f) \le M(b-a)$

$$\Rightarrow \; m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

Which shows that the numbers $L(P,f)$ and $U(P,f)$ form a bounded set.

$\Rightarrow$ The upper and lower integrals are defined for every bounded function f. ⊙

> **➢ *Riemann-Stieltjes Integral***
>
> It is a generalization of the Riemann Integral. Let $\alpha(x)$ be a monotonically increasing function on $[a,b]$. $\alpha(a)$ and $\alpha(b)$ being finite, it follows that $\alpha(x)$ is bounded on $[a,b]$. Corresponding to each partition P of $[a,b]$, we write
>
> $$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$
>
> (Difference of values of α at x_i & x_{i-1})
>
> $\because$ $\alpha(x)$ is monotonically increasing.
>
> $\therefore$ $\Delta\alpha_i \ge 0$
>
> Let f be a real function which is bounded on $[a,b]$.
>
> Put $\qquad U(P,f,\alpha) = \displaystyle\sum_{i=1}^{n} M_i \,\Delta\alpha_i$
>
> $\qquad\qquad L(P,f,\alpha) = \displaystyle\sum_{i=1}^{n} m_i \,\Delta\alpha_i$,
>
> where M_i and m_i have their usual meanings.
>
> Define
>
> $$\overline{\int_a^b} f\, d\alpha = \inf U(P,f,\alpha) \;\ldots\ldots\ldots\ldots\ldots\; (i)$$

$$\underline{\int_a^b} f\,d\alpha = \sup L(P,f,\alpha) \;\ldots\ldots\ldots\ldots\ldots\; (ii)$$

Where the infimum and supremum are taken over all partitions of $[a,b]$.

If $\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$, we denote their common value by $\int_a^b f\,d\alpha$ or $\int_a^b f(x)\,d\alpha(x)$.

This is the Riemann-Stieltjes integral or simply the Stieltjes Integral of f w.r.t. α over $[a,b]$.

If $\int_a^b f\,d\alpha$ exists, we say that f is integrable w.r.t. α, in the Riemann sense, and write $f \in R(\alpha)$.

➢ *Note*

The Riemann-integral is a special case of the Riemann-Stieltjes integral when we take $\alpha(x) = x$.

$\because$ The integral depends upon f, α, a and b but not on the variable of integration.

$\therefore$ We can omit the variable and prefer to write $\int_a^b f\,d\alpha$ instead of $\int_a^b f(x)\,d\alpha(x)$.

In the following discussion f will be assume to be real and bounded, and α monotonically increasing on $[a,b]$.

➢ *Refinement of a Partition*

Let P and P^* be two partitions of an interval $[a,b]$ such that $P \subset P^*$ i.e. every point of P is a point of P^*, then P^* is said to be a *refinement* of P.

➢ *Common Refinement*

Let P_1 and P_2 be two partitions of $[a,b]$. Then a partition P^* is said to be their *common refinement* if $P^* = P_1 \cup P_2$.

➢ *Theorem*

If P^* is a refinement of P, then

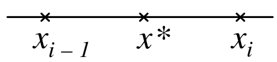
$$L(P,f,\alpha) \;\le\; L(P^*,f,\alpha) \;\ldots\ldots\ldots\ldots\ldots\; (i)$$

and
$$U(P,f,\alpha) \;\ge\; U(P^*,f,\alpha) \;\ldots\ldots\ldots\ldots\ldots\; (ii)$$

Proof

Let us suppose that P^* contains just one point x^* more than P such that $x_{i-1} < x^* < x_i$ where x_{i-1} and x_i are two consecutive points of P.

Put



$$w_1 = \inf f(x) \qquad \left(x_{i-1} \le x \le x^*\right)$$

$$w_2 = \inf f(x) \qquad \left(x^* \le x \le x_i\right)$$

It is clear that $w_1 \geq m_i$ & $w_2 \geq m_i$ where $m_i = \inf f(x)$, $\left(x_{i-1} \leq x \leq x_i\right)$.

Hence

$$\begin{aligned} L\left(P^*, f, \alpha\right) - L\left(P, f, \alpha\right) &= w_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] + w_2\left[\alpha(x_i) - \alpha(x^*)\right] \\ &\quad - m_i\left[\alpha(x_i) - \alpha(x_{i-1})\right] \\ &= w_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] + w_2\left[\alpha(x_i) - \alpha(x^*)\right] \\ &\quad - m_i\left[\alpha(x_i) - \alpha(x^*) + \alpha(x^*) - \alpha(x_{i-1})\right] \\ &= (w_1 - m_i)\left[\alpha(x^*) - \alpha(x_{i-1})\right] + (w_2 - m_i)\left[\alpha(x_i) - \alpha(x^*)\right] \end{aligned}$$

$\because$ α is a monotonically increasing function.

$\therefore$ $\alpha(x^*) - \alpha(x_{i-1}) \geq 0$, $\alpha(x_i) - \alpha(x^*) \geq 0$

$\Rightarrow$ $L\left(P^*, f, \alpha\right) - L\left(P, f, \alpha\right) \geq 0$

$\Rightarrow$ $L\left(P, f, \alpha\right) \leq L\left(P^*, f, \alpha\right)$ which is (*i*)

If P^* contains k points more than P, we repeat this reasoning k times and arrive at (*i*).

Now put

$W_1 = \sup f(x)$ $(x_{i-1} \leq x \leq x^*)$

and $W_2 = \sup f(x)$ $(x^* \leq x \leq x_i)$

Clearly $M_i \geq W_1$ & $M_i \geq W_2$

Consider

$$\begin{aligned} U\left(P, f, \alpha\right) - U\left(P^*, f, \alpha\right) &= M_i\left[\alpha(x_i) - \alpha(x_{i-1})\right] \\ &\quad - W_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] - W_2\left[\alpha(x_i) - \alpha(x^*)\right] \\ &= M_i\left[\alpha(x_i) - \alpha(x^*) + \alpha(x^*) - \alpha(x_{i-1})\right] \\ &\quad - W_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] - W_2\left[\alpha(x_i) - \alpha(x^*)\right] \\ &= \left(M_i - W_1\right)\left[\alpha(x^*) - \alpha(x_{i-1})\right] + \left(M_i - W_2\right)\left[\alpha(x_i) - \alpha(x^*)\right] \geq 0 \end{aligned}$$

$\left(\because\ \alpha \text{ is } \uparrow\right)$

$\Rightarrow$ $U\left(P, f, \alpha\right) \geq U\left(P^*, f, \alpha\right)$ which is (*ii*)

⊙

➢ ***Theorem***

Let f be a real valued function defined on $[a,b]$ and α be a monotonically increasing function on $[a,b]$. Then

$$\sup L(P,f,\alpha) \le \inf U(P,f,\alpha)$$

$$\text{i.e.} \quad \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$$

Proof

Let P^* be the common refinement of two partitions P_1 and P_2. Then

$$L(P_1,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P_2,f,\alpha)$$

Hence $L(P_1,f,\alpha) \le U(P_2,f,\alpha)$ .... (i)

If P_2 is fixed and the supremum is taken over all P_1 then (i) gives

$$\underline{\int_a^b} f\,d\alpha \le U(P_2,f,\alpha)$$

Now take the infimum over all P_2

$$\Rightarrow \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$$ ⊙

➢ ***Theorem (Condition of Integrability or Cauchy's Criterion for Integrability.)***

$f \in R(\alpha)$ on $[a,b]$ iff for every $\varepsilon > 0$ there exists a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Proof

Let $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ .... (i)

Then $L(P,f,\alpha) \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha)$

$$\Rightarrow \underline{\int_a^b} f\,d\alpha - L(P,f,\alpha) \ge 0 \quad \text{and} \quad U(P,f,\alpha) - \overline{\int_a^b} f\,d\alpha \ge 0$$

Adding these two results, we have

$$\underline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f\,d\alpha - L(P,f,\alpha) + U(P,f,\alpha) \ge 0$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \qquad \text{from } (i)$$

$$\text{i.e.} \quad 0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \varepsilon \qquad \text{for every } \varepsilon > 0.$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha \qquad \text{i.e.} \quad f \in R(\alpha)$$

Conversely, let $f \in R(\alpha)$ and let $\varepsilon > 0$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$$

Now $\overline{\int_a^b} f\,d\alpha = \inf U(P,f,\alpha)$ and $\underline{\int_a^b} f\,d\alpha = \sup L(P,f,\alpha)$

There exist partitions P_1 and P_2 such that

$$U(P_2,f,\alpha) - \int_a^b f\,d\alpha < \frac{\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ (ii)$$

and $$\int_a^b f\,d\alpha - L(P_1,f,\alpha) < \frac{\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ (iii)$$

$$U(P_2,f,\alpha) - \varepsilon/2 < \int f\,d\alpha$$
$$\int f\,d\alpha < L(P_1,f,\alpha) + \varepsilon/2$$

We choose P to be the common refinement of P_1 and P_2.
Then

$$U(P,f,\alpha) \le U(P_2,f,\alpha) < \int_a^b f\,d\alpha + \frac{\varepsilon}{2} < L(P_1,f,\alpha) + \varepsilon \le L(P,f,\alpha) + \varepsilon$$

So that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$ ⊙

➢ ***Theorem***

a) If $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ holds for some P and some ε, then it holds (with the same ε) for every refinement of P.

b) If $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ holds for $P = \{x_0, \ldots, x_n\}$ and s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |\, f(s_i) - f(t_i)\,|\,\Delta\alpha_i < \varepsilon$$

c) If $f \in R(\alpha)$ and the hypotheses of (b) holds, then

$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha\right| < \varepsilon$$

Proof

a) Let P^* be a refinement of P. Then

$$L(P,f,\alpha) \le L(P^*,f,\alpha)$$

and $$U(P^*,f,\alpha) \le U(P,f,\alpha)$$

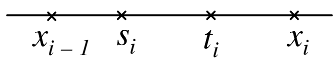
$$\Rightarrow L(P,f,\alpha) + U(P^*,f,\alpha) \le L(P^*,f,\alpha) + U(P,f,\alpha)$$

$$\Rightarrow U(P^*,f,\alpha) - L(P^*,f,\alpha) \le U(P,f,\alpha) - L(P,f,\alpha)$$

$\because\ U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$

$\therefore\ U(P^*,f,\alpha)-L(P^*,f,\alpha)<\varepsilon$

b) $P=\{x_0,....,x_n\}$ and s_i, t_i are arbitrary points in $[x_{i-1},x_i]$.



$\Rightarrow\ f(s_i)$ and $f(t_i)$ both lie in $[m_i,M_i]$.

$\Rightarrow\ |f(s_i)-f(t_i)| \le M_i-m_i$

$\Rightarrow\ |f(s_i)-f(t_i)|\Delta\alpha_i \le M_i\Delta\alpha_i-m_i\Delta\alpha_i$

$$\Rightarrow\ \sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i \le \sum_{i=1}^{n}M_i\,\Delta\alpha_i-\sum_{i=1}^{n}m_i\,\Delta\alpha_i$$

$$\Rightarrow\ \sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i \le U(P,f,\alpha)-L(P,f,\alpha)$$

$\because\ U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon$

$$\therefore\ \sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i < \varepsilon$$

c) $\because\ m_i \le f(t_i) \le M_i$

$\therefore\ \sum m_i\,\Delta\alpha_i \le \sum f(t_i)\Delta\alpha_i \le \sum M_i\,\Delta\alpha_i$

$\Rightarrow\ L(P,f,\alpha) \le \sum f(t_i)\Delta\alpha_i \le U(P,f,\alpha)$

and also $L(P,f,\alpha) \le \int_a^b f\,d\alpha \le U(P,f,\alpha)$

Using (b), we have

$$\left|\sum f(t_i)\Delta\alpha_i-\int_a^b f\,d\alpha\right| < \varepsilon$$

⊙

➢ ***Lemma***

If M & m are the supremum and infimum of f and M', m' are the supremum & infimum of $|f|$ on $[a,b]$ then $M'-m' \le M-m$.

Proof

Let $x_1,x_2\in[a,b]$, then

$$\big||f(x_1)|-|f(x_2)|\big| \le |f(x_1)-f(x_2)| \ \ldots\ldots\ldots\ldots\ldots\ldots(A)$$

$\because\ M$ and m denote the supremum and infimum of $f(x)$ on $[a,b]$

$\therefore\ f(x)\le M$ & $f(x)\ge m \quad \forall\ x\in[a,b]$

$\because\ x_1\ ,\ x_2\in[a,b]$

$\therefore\ f(x_1)\le M$ and $f(x_2)\ge m$

$\Rightarrow\ f(x_1)\le M$ and $-f(x_2)\le -m$

$\Rightarrow\ f(x_1)-f(x_2)\le M-m$ ……………... (i)

Interchanging x_1 & x_2, we get

$-[f(x_1)-f(x_2)] \le M-m$ ………….. (ii)

(i) & (ii) $\Rightarrow$ $|f(x_1)-f(x_2)| \le M-m$

$\Rightarrow$ $\big||f(x_1)|-|f(x_2)|\big| \le M-m$ by eq. (A) ………..(I)

$\because$ M' and m' denote the supremum and infimum of $|f(x)|$ on $[a,b]$

$\therefore$ $|f(x)| \le M'$ and $|f(x)| \ge m'$ $\forall$ $x\in[a,b]$

$\Rightarrow$ $\exists$ $\varepsilon>0$ such that

$$|f(x_1)| > M'-\varepsilon \text{ ………….. } (iii)$$

and $|f(x_2)| < m'+\varepsilon$ $\Rightarrow$ $-|f(x_2)|+\varepsilon > -m'$ ………….. (iv)

From (iii) and (iv), we get

$$|f(x_1)|-|f(x_2)|+\varepsilon > M'-m'-\varepsilon$$

$$\Rightarrow 2\varepsilon+|f(x_1)|-|f(x_2)| > M'-m'$$

$\because$ ε is arbitrary $\therefore$ $M'-m' \le |f(x_1)|-|f(x_2)|$ ……………. (v)

Interchanging x_1 & x_2, we get

$$M'-m' \le -\big(|f(x_1)|-|f(x_2)|\big) \text{ …………… } (vi)$$

Combining (v) and (vi), we get

$$M'-m' \le \big||f(x_1)|-|f(x_2)|\big| \text{ ……………. (II)}$$

From (I) and (II), we have the require result

$$M'-m' \le M-m$$ ⊙

➢ ***Theorem***

If $f\in R(\alpha)$ on $[a,b]$, then $|f|\in R(\alpha)$ on $[a,b]$ and $\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$.

Proof

$\because$ $f\in R(\alpha)$

$\therefore$ given $\varepsilon>0$ $\exists$ a partition P of $[a,b]$ such that

$$U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon$$

i.e. $\sum M_i\,\Delta\alpha_i-\sum m_i\,\Delta\alpha_i=\sum(M_i-m_i)\Delta\alpha_i < \varepsilon$

Where M_i and m_i are supremum and infimum of f on $[x_{i-1},x_i]$

Now if M_i' and m_i' are supremum and infimum of $|f|$ on $[x_{i-1},x_i]$ then

$$M_i'-m_i' \le M_i-m_i$$

$\Rightarrow$ $\sum(M_i'-m_i')\Delta\alpha_i \le \sum(M_i-m_i)\Delta\alpha_i$

$\Rightarrow$ $U(P,|f|,\alpha)-L(P,|f|,\alpha) \le U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon$

$\Rightarrow$ $|f|\in R(\alpha)$.

Take $c=+1$ or -1 to make $c\int f\,d\alpha \ge 0$

Then $\left|\int_a^b f\,d\alpha\right| = c\int_a^b f\,d\alpha$ ……………. (i)

Also $\quad c\,f(x) \le |f(x)| \quad \forall\ x \in [a,b]$

$$\Rightarrow \int_a^b c\,f\,d\alpha \le \int_a^b |f|\,d\alpha \Rightarrow c\int_a^b f\,d\alpha \le \int_a^b |f|\,d\alpha \ \ldots\ldots\ldots (ii)$$

From (i) and (ii), we have

$$\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha \qquad \odot$$

> ➢ ***Theorem (Ist Fundamental Theorem of Calculus)***
>
> Let $f \in R$ on $[a,b]$. For $a \le x \le b$, put $F(x) = \int_a^x f(t)\,dt$, then F is continuous on $[a,b]$; furthermore, if f is continuous at point x_0 of $[a,b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

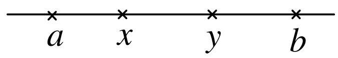
Proof

$\because\ f \in R$

$\therefore\ f$ is bounded.

Let $|f(t)| \le M$ for $t \in [a,b]$

If $a \le x < y \le b$, then



$$|F(y) - F(x)| = \left|\int_a^y f(t)\,dt - \int_a^x f(t)\,dt\right|$$

$$= \left|\int_a^x f(t)\,dt + \int_x^y f(t)\,dt - \int_a^x f(t)\,dt\right|$$

$$= \left|\int_x^y f(t)\,dt\right| \le \int_x^y |f(t)|\,dt \le M\int_x^y dt = M(y-x)$$

$$\Rightarrow |F(y) - F(x)| < \varepsilon \quad \text{for } \varepsilon > 0 \text{ provided } M|y-x| < \varepsilon$$

i.e. $\quad |F(y) - F(x)| < \varepsilon \quad$ whenever $\quad |y - x| < \dfrac{\varepsilon}{M}$

This proves the continuity (and, in fact, uniform continuity) of F on $[a,b]$.

Next, we have to prove that if f is continuous at $x_0 \in [a,b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$

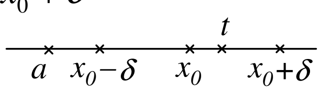
$$\text{i.e. } \lim_{t \to x_0} \frac{F(t) - F(x_0)}{t - x_0} = f(x_0)$$

Suppose f is continuous at x_0. Given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon \quad \text{if} \quad |t - x_0| < \delta \quad \text{where } t \in [a,b]$$

$$\Rightarrow f(x_0) - \varepsilon < f(t) < f(x_0) + \varepsilon \quad \text{if} \quad x_0 - \delta < t < x_0 + \delta$$

$$\Rightarrow \int_{x_0}^t \big(f(x_0) - \varepsilon\big)\,dt < \int_{x_0}^t f(t)\,dt < \int_{x_0}^t \big(f(x_0) + \varepsilon\big)\,dt$$



$$\Rightarrow \left(f(x_0)-\varepsilon\right)\int_{x_0}^{t} dt \;<\; \int_{x_0}^{t} f(t)\,dt \;<\; \left(f(x_0)+\varepsilon\right)\int_{x_0}^{t} dt$$

$$\Rightarrow \left(f(x_0)-\varepsilon\right)(t-x_0) \;<\; F(t)-F(x_0) \;<\; \left(f(x_0)+\varepsilon\right)(t-x_0)$$

$$\Rightarrow f(x_0)-\varepsilon \;<\; \frac{F(t)-F(x_0)}{t-x_0} \;<\; f(x_0)+\varepsilon$$

$$\Rightarrow \left|\frac{F(t)-F(x_0)}{t-x_0} - f(x_0)\right| < \varepsilon$$

$$\Rightarrow \lim_{t\to x_0}\frac{F(t)-F(x_0)}{t-x_0} = f(x_0)$$

$$\Rightarrow F'(x_0) = f(x_0)$$

⊙

➢ *Theorem (IInd Fundamental Theorem of Calculus)*

If $f \in R$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

Proof

$\because$ $f \in R$ on $[a,b]$

$\therefore$ given $\varepsilon > 0$, $\exists$ a partition P of $[a,b]$ such that

$$U(P,f)-L(P,f) < \varepsilon$$

$\because$ F is differentiable on $[a,b]$

$\therefore$ $\exists$ $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i)-F(x_{i-1}) = F'(t_i)\Delta x_i$$

$$\Rightarrow F(x_i)-F(x_{i-1}) = f(t_i)\Delta x_i \qquad \text{for } i = 1,2,\ldots,n \qquad \because F' = f$$

$$\Rightarrow \sum_{i=1}^{n} f(t_i)\Delta x_i = F(b)-F(a)$$

$$\Rightarrow \left| F(b)-F(a) - \int_a^b f(x)\,dx \right| < \varepsilon$$

> $\because$ if $f \in R(\alpha)$ then $\left|\sum f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha\right| < \varepsilon$

$\because$ ε is arbitrary

$$\therefore \int_a^b f(x)\,dx = F(b)-F(a)$$

⊙
