Chapter 4 – Limit and Continuity

Course Title: Real Analysis 1 Course Code: MTH321
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❖ *Limit of the function*

Suppose $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function. A number l is called the limit of f when x approaches to p if for all $\varepsilon > 0$, there exists $\delta > 0$ (depending upon ε) such that

$$|f(x) - l| < \varepsilon \text{ whenever } 0 < |x - p| < \delta.$$

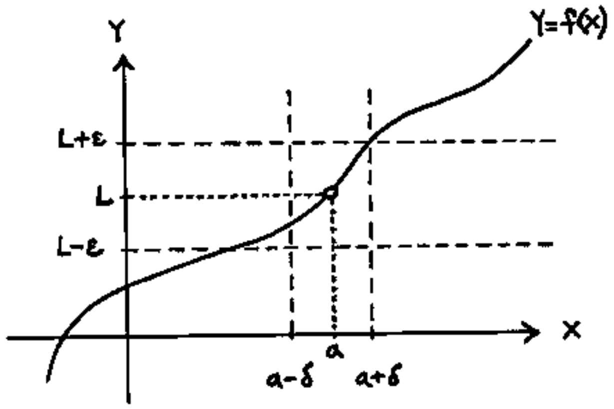
It is written as $\lim_{x \to p} f(x) = l$.

Note: *i*) It is to be noted that $p \in \mathbb{R}$ but that p need not a point of E in the above definition (p is a limit point of E which may or may not belong to E.)

ii) Even if $p \in E$, we may have $f(p) \neq \lim_{x \to p} f(x)$.

Example:

In the following diagram we have illustrated $\lim_{x \to a} f(x) = L$.



❖ *Example*

$$\lim_{x \to \infty} \frac{2x}{1+x} = 2$$

We have $\left|\dfrac{2x}{x-1} - 2\right| = \left|\dfrac{2x - 2 - 2x}{1+x}\right| = \left|\dfrac{-2}{1+x}\right| < \dfrac{2}{x}$

Now if $\varepsilon > 0$ is given we can find $\delta = \dfrac{2}{\varepsilon}$ so that

$$\left|\frac{2x}{1+x} - 2\right| < \varepsilon \quad \text{whenever} \quad x > \delta.$$ ❑

❖ *Example*

Consider the function $f(x) = \dfrac{x^2 - 1}{x - 1}$.

It is to be noted that f is not defined at $x = 1$ but if $x \neq 1$ and is very close to 1 or less then $f(x)$ equals to 2. ❑

❖ ***Definitions***

i) Let X and Y be subsets of $\mathbb{R}$, a function $f: X \to Y$ is said to tend to limit l as $x \to \infty$, if for a real number $\varepsilon > 0$ however small, $\exists$ a positive number δ which depends upon ε such that distance

$$|f(x) - l| < \varepsilon \quad \text{when } x > \delta \quad \text{and we write } \lim_{x\to\infty} f(x) = l.$$

ii) f is said to tend to a right limit l as $x \to c$ if for $\varepsilon > 0$, $\exists$ $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $c < x < c + \delta$.

And we write $\lim_{x\to c+} f(x) = l$

iii) f is said to tend to a left limit l as $x \to c$ if for $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $c - \delta < x < c$.

And we write $\lim_{x\to c-} f(x) = l$. ❑

❖ ***Example***

$$\lim_{x\to\infty} \sin\frac{1}{x} \quad \text{does not exist.}$$

Suppose that $\lim_{x\to\infty} \sin\frac{1}{x}$ exists and take it to be l, then there exist a positive real number δ such that

$$\left|\sin\frac{1}{x} - l\right| < 1 \quad \text{when} \quad 0 < |x - 0| < \delta \quad (\text{we take } \varepsilon = 1 > 0 \text{ here})$$

We can find a positive integer n such that

$$\frac{2}{n\pi} < \delta \quad \text{then} \quad \frac{2}{(4n+1)\pi} < \delta \quad \text{and} \quad \frac{2}{(4n+3)\pi} < \delta$$

It thus follows

$$\left|\sin\frac{(4n+1)\pi}{2} - l\right| < 1 \quad \Rightarrow |1 - l| < 1$$

$$\text{and} \quad \left|\sin\frac{(4n+3)\pi}{2} - l\right| < 1 \quad \Rightarrow |-1 - l| < 1 \quad \text{or} \quad |1 + l| < 1$$

So that

$$2 = |1 + l + 1 - l| \le |1 + l| + |1 - l| < 1 + 1 \quad \Rightarrow 2 < 2$$

This is impossible; hence limit of the function does not exist. ❑

❖ ***Example***

Consider the function $f: [0,1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & \textit{if } x \textit{ is rational} \\ 1 & \textit{if } x \textit{ is irratioanl} \end{cases}$$

Show that $\lim_{x\to p} f(x)$ where $p \in [0,1]$ does not exist.

Solution

Let $\lim_{x\to p} f(x)=q$, if given $\varepsilon>0$ we can find $\delta>0$ such that

$$|f(x)-q|<\varepsilon \quad \text{whenever} \quad |x-p|<\delta.$$

Consider the irrational $(r-s,r+s)\subset[0,1]$ such that r is rational and s is irrational.

Then $f(r)=0$ & $f(s)=1$

Suppose $\lim_{x\to p} f(x)=q$ then

$$|f(s)|=1$$

$$\Rightarrow\ 1=|f(s)-q+q|$$

$$=|(f(s)-q+q-0|$$

$$=|f(s)-q+q-f(r)| \qquad \because\ 0=f(r)$$

$$\le|f(s)-q|+|f(r)-q|<\varepsilon+\varepsilon$$

i.e. $1<\varepsilon+\varepsilon$

$$\Rightarrow\ 1<\frac{1}{4}+\frac{1}{4} \qquad \text{if } \varepsilon=\frac{1}{4}$$

Which is absurd.

Hence the limit of the function does not exist. ❑

❖ *Theorem*

If $\lim_{x\to c} f(x)$ exists then it is unique.

Proof

Suppose $\lim_{x\to c} f(x)$ is not unique.

Take $\lim_{x\to c} f(x)=l_1$ and $\lim_{x\to c} f(x)=l_2$ where $l_1\neq l_2$.

$\Rightarrow\ \exists$ real numbers δ_1 and δ_2 such that

$$|f(x)-l_1|<\varepsilon \quad \text{whenever} \quad |x-c|<\delta_1$$

$$\& \quad |f(x)-l_2|<\varepsilon \quad \text{whenever} \quad |x-c|<\delta_2$$

Now $|l_1-l_2|=\left|\left(f(x)-l_1\right)-\left(f(x)-l_2\right)\right|$

$$\le|f(x)-l_1|+|f(x)-l_2|$$

$$<\varepsilon+\varepsilon \quad \text{whenever} \quad |x-c|<\min(\delta_1,\delta_2)$$

$$\Rightarrow\ l_1=l_2$$

❑

❖ *Theorem*

Let $f:E\to\mathbb{R}$ and $g:E\to\mathbb{R}$ be real valued functions. If $\lim_{x\to p} f(x)=A$ and $\lim_{x\to p} g(x)=B$ then

i- $\lim_{x\to p}\left(f(x)\pm g(x)\right)=A\pm B$,

ii- $\lim_{x\to p}(fg)(x)=AB$,

iii- $\lim_{x\to p}\left(\frac{f(x)}{g(x)}\right)=\frac{A}{B}$ provided $B\neq 0$.

❖ *Continuity*

Suppose $E\subseteq\mathbb{R}$ and $f:E\to\mathbb{R}$ be a function. Then f is said to be continuous at p if for every $\varepsilon>0$ $\exists$ a $\delta>0$ such that

$$|f(x)-f(p)|<\varepsilon \text{ for all points } x\in E \text{ for which } 0<|x-p|<\delta.$$

Note:

(i) If f is continuous at every point of E. Then f is said to be continuous on E.

(ii) It is to be noted that f has to be defined at p iff $\lim_{x\to p} f(x)=f(p)$. ❑

❖ *Examples*

$f(x)=x^2$ is continuous $\forall$ $x\in\mathbb{R}$.

Here $f(x)=x^2$, Take $p\in\mathbb{R}$

Then $|f(x)-f(p)|<\varepsilon$

$\Rightarrow |x^2-p^2|<\varepsilon$

$\Rightarrow |(x-p)(x+p)|<\varepsilon$

$\Rightarrow |x-p|<\varepsilon=\delta$

Since p is arbitrary real number,

therefore the function $f(x)$ is continuous $\forall$ real numbers. ❑

❖ *Example*

$f(x)=\sqrt{x}$ is continuous on $[0,\infty[$.

Let c be an arbitrary point such that $0<c<\infty$

For $\varepsilon>0$, we have

$$|f(x)-f(c)|=|\sqrt{x}-\sqrt{c}|=\frac{|x-c|}{\sqrt{x}+\sqrt{c}}<\frac{|x-c|}{\sqrt{c}}$$

$$\Rightarrow |f(x)-f(c)|<\varepsilon \quad \text{whenever} \quad \frac{|x-c|}{\sqrt{c}}<\varepsilon$$

i.e. $|x-c|<\sqrt{c}\,\varepsilon=\delta$

$\Rightarrow$ f is continuous for $x=c$.

$\because$ c is an arbitrary point lying in $[0,\infty[$

$\therefore$ $f(x)=\sqrt{x}$ is continuous on $[0,\infty[$ ❑

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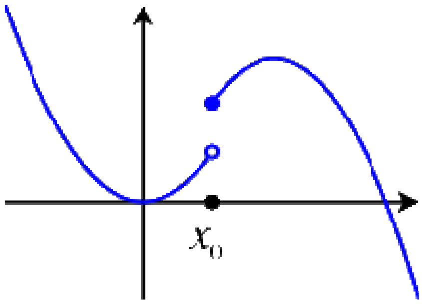
❖ *Right continuous and left continuous*

Let f be a real valued function. It is said to be right continuous at point a if $\lim_{x\to a+} f(x) = f(a)$ and it is said to be left continuous at point a if $\lim_{x\to a-} f(x) = f(a)$.

❖ *Example*



Consider a function given in above graph. We see f is not continuous at point x_0. It is right continuous at point x_0 but not left continuous at point x_0.

❖ *Continuity at closed interval*

A function $f:[a,b]\to\mathbb{R}$ is said to be continuous on closed interval $[a,b]$ if

(i) f is continuous on (a,b)

(ii) f is right continuous at a.

(iii) f is left continuous at b.

❖ *Theorem (The intermediate value theorem)*

Suppose f is continuous on $[a,b]$ and $f(a)\neq f(b)$, then given a number λ that lies between $f(a)$ and $f(b)$, $\exists$ a point c, $a<c<b$ with $f(c)=\lambda$.

Proof

Let $f(a)<f(b)$ and $f(a)<\lambda<f(b)$.

Suppose $g(x)=f(x)-\lambda$

Then $g(a)=f(a)-\lambda<0$ and $g(b)=f(b)-\lambda>0$

$\Rightarrow\ \exists$ a point c between a and b such that $g(c)=0$

$$\Rightarrow\ f(c)-\lambda=0 \quad \Rightarrow\ f(c)=\lambda$$

If $f(a)>f(b)$ then take $g(x)=\lambda-f(x)$ to obtain the required result. ❑

❖ *Uniform continuity*

Suppose $f:E\to\mathbb{R}$ is a real valued function. We say that f is uniformly continuous on E if for every $\varepsilon>0$ there exists $\delta>0$ such that

$$|f(p)-f(q)|<\varepsilon \quad \forall \quad p,q\in E \quad \text{for which} \quad |p-q|<\delta$$

The uniform continuity is a property of a function on a set i.e. it is a global property but continuity can be defined at a single point i.e. it is a local property. Uniform continuity of a function at a point has no meaning.

It is evident that every uniformly continuous function is continuous.

To emphasize a difference between continuity and uniform continuity on set S, we consider the following examples. ❑

❖ *Example*

Let S be a half open interval $0 < x \le 1$ and let f be defined for each x in S by the formula $f(x) = x^2$. It is uniformly continuous on S. To prove this observe that we have

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| \\ &= |x-y||x+y| \\ &< 2|x-y| \end{aligned}$$

If $|x-y| < \delta$ then $|f(x) - f(y)| < 2\delta = \varepsilon$

Hence if ε is given we need only to take $\delta = \dfrac{\varepsilon}{2}$ to guarantee that

$$|f(x) - f(y)| < \varepsilon \text{ for every pair } x, y \text{ with } |x-y| < \delta$$

Thus f is uniformly continuous on the set S. ❑

❖ *Example*

Let S be the half open interval $0 < x \le 1$ and let a function f be defined for each x in S by the formula $f(x) = \dfrac{1}{x}$. This function is continuous on the set S, however we shall prove that this function is not uniformly continuous on S.

Solution

Let suppose $\varepsilon = 10$ and suppose we can find a δ, $0 < \delta < 1$, to satisfy the condition of the definition.

Taking $x = \delta$, $y = \dfrac{\delta}{11}$, we obtain

$$|x - y| = \frac{10\delta}{11} < \delta$$

and

$$|f(x) - f(y)| = \left| \frac{1}{\delta} - \frac{11}{\delta} \right| = \frac{10}{\delta} > 10$$

Hence for these two points we have $|f(x) - f(y)| > 10$.

Which contradict the definition of uniform continuity.

Hence the given function being continuous on a set S is not uniformly continuous on S. ❑