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Mathematics Class 9th

Chapter No: 4 **Algebraic Expression and Algebraic Formulas**

Algebraic Expression:

An expression which connects variables and constants by algebraic operations of addition, subtraction, multiplication and division is called an algebraic expression.

A few algebraic expressions are given below:

i. 14
ii. $x + 2y$
iii. $4x - y + 5$
iv. $5x^2 - 4x$

Polynomial:

A polynomial is an algebraic expression consisting of one or more terms in each of which the exponent of the variable is zero or a positive integer.

For Example:

i. 13
ii. $-x$
iii. $5x + 3y$
iv. $x^2 - 3x + 1$

The following algebraic expressions are not polynomials:

i. x^{-2}
ii. $x^2 - y^{-4}$
iii. $\frac{1}{y}$

Degree of the Polynomial:

Degree of the polynomial is the degree of the highest degree of a part(term) in a polynomial.

For Example:

i. $x + 1$ (Polynomial having degree 1)
ii. $x^2 + x$ (Polynomial having degree 2)
iii. $x^3 + xy - 1$ (Polynomial having degree 2)

iv. $x^2y^2 + x^3 + y^2 - 5$ (Polynomial having degree 4 because term x^2y^2 = sum of the exponents is equal to 4)
v. $2x^3y^2$ (Polynomial having degree 5)
vi. $2.3 + 0.2x$ (Polynomial having degree 1)
vii.

Note

Degree of a term in a polynomial is the sum of the exponents on the variable in a single term.

For Example:

i. The degree of $2x^3y^4$ is 4 as $3 + 4 = 7$.
ii. The degree of $x^2y + x^4$ is 4.

Rational Expression:

The quotient $\frac{\mathrm{p(x)}}{\mathrm{q(x)}}$ of two polynomials $\mathrm{p(x)}$ and $\mathrm{q(x)}$, where $\mathrm{q(x)} \neq 0$ is called rational expression.

For Example:

i. $\frac{3x+1}{5x+4}$, $5x + 4 \neq 0$.

In the rational expression $\frac{\mathrm{p(x)}}{\mathrm{q(x)}}$, $\mathrm{p(x)}$ is called the numerator and $\mathrm{q(x)}$ is called the denominator of the rational expression.

Note

Every polynomial $\mathrm{p(x)}$ is a rational expression, because we can write $\mathrm{p(x)}$ as $\frac{\mathrm{p(x)}}{1}$. But every rational expression not be a polynomial.

Rational Expression in its Lowest Form:

The rational expression $\frac{\mathrm{p(x)}}{\mathrm{q(x)}}$ is said to be in its lowest forms if $\mathrm{p(x)}$ and $\mathrm{q(x)}$ are polynomials with integral coefficients and have no common factor.

For Example:

i. $\frac{x-1}{x+1}$ ii. $\frac{x^2-3}{x+2}$ both rational expressions are in its lowest form.

To Check Whether a Rational Expression is in Lowest Form:

To check the rational expression $\frac{\mathrm{p(x)}}{\mathrm{q(x)}}$, find the H. C. F of p(x) and q(x). If H. C. F is 1 then the rational expression is in its lowest form.

Reduce the Rational Expression to its Lowest Form:

A rational expression can be reduced to its lowest forms by first factorizing both the polynomials in the numerator and denominator and then canceling the common factors between them.

Example: Reduce $\frac{3x^2+18x+27}{5x^2-45}$ to their lowest form.

Solution:

$$\frac{3x^2+18x+27}{5x^2-45}$$

$$=\frac{3[x^2+6x+9]}{5[x^2-9]}$$

$$=\frac{3[x^2+6x+9]}{5[(x)^2-(3)^2]}$$

$$=\frac{3[(x)^2+2(x)(3)+(3)^2]}{5[(x)^2-(3)^2]}$$

$$=\frac{3(x+3)^2}{5(x+3)(x-3)}$$

$$=\frac{3(x+3)(x+3)}{5(x+3)(x-3)}$$

$$=\frac{3(x+3)}{5(x-3)}$$

Remember!

$$(a+b)^2=a^2+2ab+b^2$$

$$a^2-b^2=(a+b)(a-b)$$

Sum and Difference of Rational Expression:

For finding the sum and difference of algebraic expression containing rational expression, we take the L. C. M of the denominator and simplifying them.

Example: Simplify $\frac{1}{x-y}-\frac{1}{x+y}+\frac{2}{x^2-y^2}$

Solution:

$$\frac{1}{x-y}-\frac{1}{x+y}+\frac{2}{x^2-y^2}=\frac{1}{x-y}-\frac{1}{x+y}+\frac{2}{(x+y)(x-y)}$$

Taking L.C.M

$$=\frac{(x+y)-(x-y)+2x}{(x+y)(x-y)}$$

$$=\frac{x+y-x+y+2x}{(x+y)(x-y)}$$

$$=\frac{2x+2y}{(x+y)(x-y)}$$

$$=\frac{2(x+y)}{(x+y)(x-y)}$$ Taking '2' as a Common

$$=\frac{2}{(x-y)}$$ Cancelling "$(x+y)$"

Product of the Rational Expressions:

Product of rational expressions is explained through example:

Example: Find the product $\frac{x+2}{2x-3y}\cdot\frac{4x^2-9y^2}{xy+2y}$

Solution:

$$\frac{x+2}{2x-3y}\cdot\frac{4x^2-9y^2}{xy+2y}=\frac{(x+2)(4x^2-9y^2)}{(2x-3y)(xy+2y)}$$

$$=\frac{(x+2)[(2x)^2-(3y)^2]}{(2x-3y)(xy+2y)}$$ $\because a^2-b^2=(a+b)(a-b)$

$$=\frac{(x+2)(2x+3y)(2x-3y)}{(2x-3y)(xy+2y)}$$ Cancelling "$(2x-3y)$"

$$=\frac{(x+2)(2x+3y)}{y(x+2)}$$ Cancelling "$(x+2)$"

$$=\frac{(2x+3y)}{y}$$

Division of Rational Expression:

In order to divide one rational expression with another we first invert for changing division to multiplication and simplifying the resulting product to its lowest term.

Example: Simplify $\dfrac{7xy}{x^2-4x+4} \div \dfrac{14y}{x^2-4}$

Solution:

$$\frac{7xy}{x^2-4x+4} \div \frac{14y}{x^2-4} = \frac{7xy}{x^2-4x+4} \times \frac{x^2-4}{14y}$$

$$= \frac{7xy}{(x)^2-2(x)(2)+(2)^2} \times \frac{(x)^2-(2)^2}{14y}$$

$$= \frac{7xy}{(x-2)^2} \times \frac{(x+2)(x-2)}{14y}$$

$$= \frac{7xy}{(x-2)(x-2)} \times \frac{(x+2)(x-2)}{14y}$$

$$= \frac{x}{(x-2)} \times \frac{(x+2)}{2}$$

$$= \frac{x(x+2)}{2(x-2)}$$

Exercise 4.1

1. **Identify whether the following algebraic expressions are polynomials (Yes or No).**

   i. $3x^2 + \frac{1}{x} - 5$ — No, (Because Negative Exponent $\frac{1}{x} = x^{-1}$)

   ii. $3x^3 - 4x^2 - x\sqrt{x} + 3$ — No, (Because due to $\sqrt{x}$)

   iii. $x^2 - 3x + \sqrt{2}$ — Yes

   iv. $\frac{3x}{2x-1} + 8$ — No, (Because Negative Exponent)

2. **State whether each of the following expressions is a rational expressions or not.**

   i. $\frac{3\sqrt{3}}{3\sqrt{x}+5}$ — No, (Because given expressions are not Polynomials)

   ii. $\frac{x^3 - 2x^2 + \sqrt{3}}{2 + 3x + x^2}$ — Yes

   iii. $\frac{x^2 + 6x + 9}{x^2 - 9}$ — Yes

   iv. $\frac{2\sqrt{3} + 3}{2\sqrt{x} - 3}$ — No, (Because given expressions are not Polynomials)

3. Reduce the following rational expressions to the lowest form.

i. $\frac{120x^2y^3z^5}{30x^3yz^2}$

Solution:

$$\frac{120x^2y^3z^5}{30x^3yz^2} = \frac{120}{30} \times \frac{x^2y^3z^5}{x^3yz^2}$$

$$= 4x^2y^3z^5 \times x^{-3}y^{-1}z^{-2}$$

$$= 4x^2x^{-3}y^3y^{-1}z^5z^{-2}$$ $\because$ Adding Like Terms

$$= 4x^{-1}y^2z^3$$

$$= \frac{4y^2z^3}{x}$$

ii. $\frac{8a(x+1)}{2(x^2-1)}$

Solution:

$$\frac{8a(x+1)}{2(x^2-1)} = \frac{8a}{2} \times \frac{(x+1)}{(x^2-1)}$$

$$= \frac{4a(x+1)}{(x^2-1)}$$

$$= \frac{4a(x+1)}{(x)^2-(1)^2}$$

$$= \frac{4a(x+1)}{(x-1)(x+1)}$$ $\because$ Cancelling $'x+1'$

$$= \frac{4a}{(x-1)}$$

iii. $\frac{(x+y)^2-4xy}{(x-y)^2}$

Solution:

$$\frac{(x+y)^2-4xy}{(x-y)^2} = \frac{x^2+y^2+2xy-4xy}{x^2+y^2-2xy}$$

$$= \frac{x^2 + y^2 - 2xy}{x^2 + y^2 - 2xy} = 1$$

iv. $\frac{(x^3 - y^3)(x^2 - 2xy + y^2)}{(x - y)(x^2 + xy + y^2)}$

Solution:

$$\frac{(x^3 - y^3)(x^2 - 2xy + y^2)}{(x - y)(x^2 + xy + y^2)}$$

$$= \frac{(x^3 - y^3)(x^2 - 2xy + y^2)}{(x^3 - y^3)} \qquad \because a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$= (x^2 - 2xy + y^2)$$

$$= (x)^2 - 2(x)(y) + (y)^2$$

$$= (x - y)^2$$

v. $\frac{(x + 2)(x^2 - 1)}{(x + 1)(x^2 - 4)}$

Solution:

$$\frac{(x + 2)(x^2 - 1)}{(x + 1)(x^2 - 4)} = \frac{(x + 2)[(x)^2 - (1)^2]}{(x + 1)[(x)^2 - (2)^2]}$$

$$= \frac{(x + 2)(x + 1)(x - 1)}{(x + 1)(x + 2)(x - 2)} = \frac{(x - 1)}{(x - 2)}$$

vi. $\frac{x^2 - 4x + 4}{2x^2 - 8}$

Solution:

$$\frac{x^2 - 4x + 4}{2x^2 - 8} = \frac{(x)^2 - 2(x)(2) + (2)^2}{2(x^2 - 4)}$$

$$= \frac{(x - 2)^2}{2[(x)^2 - (2)^2]} = \frac{(x - 2)(x - 2)}{2(x + 2)(x - 2)}$$

$$= \frac{(x - 2)}{2(x + 2)}$$

vii. $\frac{64x^5 - 64x}{(8x^2 + 8)(2x + 2)}$

Solution:

$$\frac{64x^5-64x}{(8x^2+8)(2x+2)}=\frac{64x(x^4-1)}{8(x^2+1)2(x+1)}$$

$$=\frac{64x[(x^2)^2-(1)^2]}{16(x^2+1)(x+1)}=\frac{64x(x^2+1)(x^2-1)}{16(x^2+1)(x+1)}$$

$$=\frac{64x}{16}\times\frac{(x^2+1)(x^2-1)}{(x^2+1)(x+1)}=\frac{4x(x^2-1)}{(x+1)}$$

$$=\frac{4x[(x)^2-(1)^2]}{(x+1)}=\frac{4x(x+1)(x-1)}{(x+1)}$$

$$=4x(x-1)$$

viii. $\frac{9x^2-(x^2-4)^2}{4+3x-x^2}$

Solution:

$$\frac{9x^2-(x^2-4)^2}{4+3x-x^2}=\frac{(3x)^2-(x^2-4)^2}{4+3x-x^2}$$

$$=\frac{(3x+x^2-4)[3x-(x^2-4)]}{(3x-x^2+4)}=\frac{(3x+x^2-4)(3x-x^2+4)}{(3x-x^2+4)}$$

$$=3x+x^2-4$$

4. Evaluate: $\frac{x^3y-2z}{xz}$ for,

i. $x=3, y=-1, z=-2$

Solution:

$$\frac{x^3y-2z}{xz}=\frac{(3)^3(-1)-2(-2)}{(3)(-2)}=\frac{-27+4}{-6}=\frac{-23}{-6}=\frac{23}{6}=3\frac{5}{6}$$

ii. $x=-1, y=-9, z=4$

Solution:

$$\frac{x^3y-2z}{xz}=\frac{(-1)^3(-9)-2(4)}{(-1)(4)}=\frac{-1(-9)-8}{-4}=\frac{9-8}{-4}=-\frac{1}{4}$$

Evaluate: $\frac{x^2y^3-5z^4}{xyz}$ for $x=4, y=-2, z=-1$

Solution:

$$\frac{x^2y^3-5z^4}{xyz}=\frac{(4)^2(-2)^3-5(-1)^4}{(4)(-2)(-1)}=\frac{16(-8)-5(1)}{8}$$

$$=\frac{-128-5}{8}=\frac{-133}{8}=-16\frac{5}{8}$$

5. Perform the indicated operations and simplify.

i. $\dfrac{\mathbf{15}}{\mathbf{2x-3y}}-\dfrac{\mathbf{4}}{\mathbf{3y-2x}}$

Solution:

$$\frac{15}{2x-3y}-\frac{4}{3y-2x}=\frac{15}{2x-3y}-\frac{4}{-(2x-3y)}$$

$$=\frac{15}{2x-3y}+\frac{4}{(2x-3y)}=\frac{15+4}{2x-3y}=\frac{19}{2x-3y}$$

ii. $\dfrac{\mathbf{1+2x}}{\mathbf{1-2x}}-\dfrac{\mathbf{1-2x}}{\mathbf{1+2x}}$

Solution:

$$\frac{1+2x}{1-2x}-\frac{1-2x}{1+2x}=\frac{(1+2x)^2-(1-2x)^2}{(1-2x)(1+2x)}$$

$$=\frac{(1+4x+4x^2)-(1-4x+4x^2)}{(1)^2-(2x)^2}=\frac{1+4x+4x^2-1+4x-4x^2}{1-4x^2}$$

$$=\frac{8x}{1-4x^2}$$

iii. $\dfrac{\mathbf{x^2-25}}{\mathbf{x^2-36}}-\dfrac{\mathbf{x+5}}{\mathbf{x+6}}$

Solution:

$$\frac{x^2-25}{x^2-36}-\frac{x+5}{x+6}$$

$$=\frac{(x^2-25)-(x+5)(x-6)}{x^2-36}$$

$$=\frac{[(x)^2-(5)^2]-(x+5)(x-6)}{x^2-36}$$

Note

$$\frac{x^2-36}{x+6}=\frac{(x)^2-(6)^2}{x+6}=\frac{(x+6)(x-6)}{x+6}=(x-6)$$

$$=\frac{(x+5)(x-5)-(x+5)(x-6)}{x^2-36}$$

$$=\frac{(x+5)[(x-5)-(x-6)]}{x^2-36}$$ ∵ Taking (x + 5) as a Common

$$=\frac{(x+5)(x-5-x+6)}{x^2-36}$$

$$=\frac{(x+5)(1)}{x^2-36}$$

$$=\frac{(x+5)}{x^2-36}$$

iv. $\frac{x}{x-y}-\frac{y}{x+y}-\frac{2xy}{x^2-y^2}$

Solution:

$$\frac{x}{x-y}-\frac{y}{x+y}-\frac{2xy}{x^2-y^2}$$

$$=\frac{x(x+y)-y(x-y)-2xy}{x^2-y^2}$$

$$=\frac{x^2+xy-xy+y^2-2xy}{x^2-y^2}$$

$$=\frac{x^2+y^2-2xy}{x^2-y^2}$$

$$=\frac{(x-y)^2}{x^2-y^2}$$

$$=\frac{(x-y)(x-y)}{(x)^2-(y)^2}$$

$$=\frac{(x-y)(x-y)}{(x-y)(x+y)}$$

$$=\frac{(x-y)}{(x+y)}$$

v. $\frac{x-2}{x^2+6x+9}-\frac{x+2}{2x^2-18}$

Solution:

$$\frac{x-2}{x^2+6x+9}-\frac{x+2}{2x^2-18}$$

$$=\frac{x-2}{(x)^2+2(x)(3)+(3)^2}-\frac{x+2}{2[x^2-9]}$$

$$=\frac{x-2}{(x+3)^2}-\frac{x+2}{2[(x)^2-(3)^2]}$$

$$=\frac{x-2}{(x+3)^2}-\frac{x+2}{2(x+3)(x-3)}$$

$$=\frac{2(x-3)(x-2)-(x+3)(x+2)}{2(x-3)(x+3)^2}$$

$$=\frac{2[x(x-2)-3(x-2)]-[x(x+2)+3(x+2)]}{2(x-3)(x+3)^2}$$

$$=\frac{2(x^2-2x-3x+6)-(x^2+2x+3x+6)}{2(x-3)(x+3)^2}$$

$$=\frac{2(x^2-5x+6)-(x^2+5x+6)}{2(x-3)(x+3)^2}$$

$$=\frac{2x^2-10x+12-x^2-5x-6}{2(x-3)(x+3)^2}$$

$$=\frac{x^2-15x+6}{2(x-3)(x+3)^2}$$

vi. $\mathbf{\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{x^2+1}-\frac{4}{x^4-1}}$

Solution:

$$\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{x^2+1}-\frac{4}{x^4-1}$$

$$=\frac{(x+1)(x^2+1)-(x-1)(x^2+1)-2(x^2-1)-4}{x^4-1}$$

$$=\frac{[x(x^2+1)+1(x^2+1)]-[x(x^2+1)-(x^2+1)]-2(x^2-1)-4}{x^4-1}$$

$$=\frac{(x^3+x+x^2+1)-(x^3+x-x^2-1)-2x^2+2-4}{x^4-1}$$

Note
$x^4-1=(x^2)^2-(1)^2=(x^2+1)(x^2-1)$ $=(x^2+1)[(x)^2-(1)^2]$ $=(x^2+1)(x+1)(x-1)$

$$=\frac{x^3+x+x^2+1-x^3-x+x^2+1-2x^2+2-4}{x^4-1}$$

$$=\frac{0}{x^4-1}=0$$

6. Perform the indicated operations and simplify.

i. $(x^2-49)\dfrac{5x+2}{x+7}$

Solution:

$$(x^2-49)\frac{5x+2}{x+7}=(x)^2-(7)^2\frac{5x+2}{x+7}$$

$$=(x+7)(x-7)\frac{5x+2}{x+7}$$

$$=(x-7)(5x+2)$$

ii. $\dfrac{4x-12}{x^2-9}\div\dfrac{18-2x^2}{x^2+6x+9}$

Solution:

$$\frac{4x-12}{x^2-9}\div\frac{18-2x^2}{x^2+6x+9}$$

$$=\frac{4(x-3)}{x^2-9}\div\frac{2(9-x^2)}{(x)^2+2(x)(3)+(3)^2}$$

$$=\frac{4(x-3)}{(x)^2-(3)^2}\div\frac{2[(3)^2-(x)^2]}{(x+3)^2}$$

$$=\frac{4(x-3)}{(x+3)(x-3)}\div\frac{2(3+x)(3-x)}{(x+3)(x+3)}$$

$$=\frac{4(x-3)}{(x+3)(x-3)}\times\frac{(x+3)(x+3)}{2(3+x)(3-x)}$$

$$=\frac{2}{(3-x)}$$

Note

$(x+3)=(3+x)$

iii. $\dfrac{x^6-y^6}{x^2-y^2}\div x^4+x^2y^2+y^4$

Solution:

$$\frac{x^6 - y^6}{x^2 - y^2} \div x^4 + x^2y^2 + y^4$$

$$= \frac{\left(x^2\right)^3 - \left(x^2\right)^3}{(x^2 - y^2)} \div x^4 + x^2y^2 + y^4$$

$$= \frac{(x^2 - y^2)(x^2 + x^2y^2 + y^2)}{(x^2 - y^2)} \div x^4 + x^2y^2 + y^4$$

$$= \frac{(x^2 - y^2)(x^2 + x^2y^2 + y^2)}{(x^2 - y^2)} \times \frac{1}{(x^2 + x^2y^2 + y^2)}$$

$$= 1$$

iv. $\boldsymbol{\frac{x^2 - 1}{x^2 + 2x + 1} \cdot \frac{x + 5}{1 - x}}$

Solution:

$$\frac{x^2 - 1}{x^2 + 2x + 1} \cdot \frac{x + 5}{1 - x}$$

$$= \frac{(x)^2 - (1)^2}{(x)^2 + 2(x)(1) + (1)^2} \cdot \frac{x + 5}{1 - x}$$

$$= \frac{(x + 1)(x - 1)}{(x + 1)^2} \cdot \frac{x + 5}{1 - x}$$

$$= \frac{(x + 1)(x - 1)}{(x + 1)(x + 1)} \cdot \frac{x + 5}{-(x - 1)}$$

$$= -\frac{x + 5}{x + 1}$$

v. $\boldsymbol{\frac{x^2 - xy}{y(x - y)} \cdot \frac{x^2 + xy}{y(x + y)} \div \frac{x^2 - x}{xy - 2y}}$

Solution:

$$\frac{x^2 - xy}{y(x - y)} \cdot \frac{x^2 + xy}{y(x + y)} \div \frac{x^2 - x}{xy - 2y}$$

$$= \frac{x(x - y)}{y(x - y)} \cdot \frac{x(x + y)}{y(x + y)} \div \frac{x(x - 1)}{y(x - 2)}$$

$$= \frac{x(x - y)}{y(x - y)} \cdot \frac{x(x + y)}{y(x + y)} \times \frac{y(x - 2)}{x(x - 1)} = \frac{x(x - 2)}{y(x - 1)}$$

Algebraic Formulae

1. $(a+b)^2+(a-b)^2=2(a^2+b^2)$

Example: If $a+b=7$ and $a-b=3$ then find the value of a^2+b^2.

Solution:

Given:

$a+b=7$ and $a-b=3$

To find the value of a^2+b^2, we use the above formula

$(a+b)^2+(a-b)^2=2(a^2+b^2)$

Now by putting the values of $a+b=7$ and $a-b=3$, we get

$(7)^2+(3)^2=2(a^2+b^2)$

$49+9=2(a^2+b^2)$

$58=2(a^2+b^2)$

$29=a^2+b^2$

$a^2+b^2=29$

2. $(a+b)^2-(a-b)^2=4ab$

Example: If $a+b=7$ and $a-b=3$ then find the value of ab.

Solution:

Given:

$a+b=7$ and $a-b=3$

To find the value of a^2+b^2, we use the above formula

$(a+b)^2-(a-b)^2=4ab$

Now by putting the values of $a+b=7$ and $a-b=3$, we get

$(7)^2-(3)^2=4ab$

$49-9=4ab$

$40=4ab$

$10=ab$ or

$ab=10$

3. $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

$= a^2+b^2+c^2+2(ab+bc+ca)$

Example: If $a^2+b^2+c^2=43$ and $ab+bc+ca=3$, then find the value of $a+b+c$.

Solution:

Given:

$a^2+^2+c^2=43$ and $ab+bc+ca=3$

To find the value of $a+b+c$, we use the above formula

$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

Now by putting the values of $a^2+^2+c^2=43$ and $ab+bc+ca=3$, we get

$(a+b+c)^2 = 43+2(3)$

$(a+b+c)^2 = 43+6$

$(a+b+c)^2 = 49$

Taking square root on both sides

$\sqrt{(a+b+c)^2} = \sqrt{49}$

$a+b+c=7$

Example: If $a+b+c=7$ and $ab+bc+ca=9$, then find the value of $a^2+b^2+c^2$.

Solution:

Given:

$a+b+c=7$ and $ab+bc+ca=9$

To find the value of $a^2+b^2+c^2$, we use the formula

$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

Now by putting the values of $a+b+c=7$ and $ab+bc+ca=9$, we get

$(7)^2 = a^2+b^2+c^2+2(9)$

$49 = a^2+b^2+c^2+18$

$a^2+b^2+c^2 = 49-18$

$a^2+b^2+c^2 = 31$

4. **Cubic Formulas:**
    i. $(\boldsymbol{a}+\boldsymbol{b})^3=\boldsymbol{a}^3+3\boldsymbol{ab}(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{b}^3$
    ii. $(\boldsymbol{a}-\boldsymbol{b})^3=\boldsymbol{a}^3-3\boldsymbol{ab}(\boldsymbol{a}-\boldsymbol{b})-\boldsymbol{b}^3$

Example: If $2x-3y=10$ and $xy=2$, then find the value of $8x^3-27y^3$.

Solution:

Given:

$2x-3y=10$ .... (1)

To find the value of $8x^3-27y^3$, we use the formula

$(a-b)^3=a^3-3ab(a-b)-b^3$

Taking cube on both slides of eq. (1)

$(2x-3y)^3=(10)^3$

$(2x)^3-3(2x)(3y)(2x-3y)-(3y)^3=1000$

$8x^3-18xy(2x-3y)-27y^3=1000$

Now put the value of $xy=2$ and $2x-3y=10$ given question

$8x^3-18(2)(10)-27y^3=1000$

$8x^3-360-27y^3=1000$

$8x^3-27y^3=1000+360$

$8x^3-27y^3=1360$

Example: If $x+\frac{1}{x}=8$, then find the value of $x^3+\frac{1}{x^3}$

Solution:

$x+\frac{1}{x}=8$ .... (1)

Taking cube on both sides of the above equation

$\left(x+\frac{1}{x}\right)^3=(8)^3$

$$x^3 + 3(x)\left(\frac{1}{x}\right)\left(x + \frac{1}{x}\right) + \left(\frac{1}{x}\right)^3 = 512$$

$$x^3 + 3\left(x + \frac{1}{x}\right) + \frac{1}{x^3} = 512$$

Now put the value of $x + \frac{1}{x} = 8$

$$x^3 + 3(8) + \frac{1}{x^3} = 512$$

$$x^3 + \frac{1}{x^3} + 24 = 512$$

$$x^3 + \frac{1}{x^3} = 512 - 24$$

$$x^3 + \frac{1}{x^3} = 488$$

5. More Cubic Formulas

i. $\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$

ii. $\boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$

Example: Factorize $\boldsymbol{64x^3 + 343y^3}$

Solution:

$$64x^3 + 343y^3$$

$$= (4x)^3 + (7y)^3$$

$$= (4x + 7y)[(4x)^2 - (4x)(7y) + (7y)^2]$$

$$= (4x + 7y)(16x^2 - 28xy + 49y^2)$$

Example: Find the product of $\boldsymbol{(x + y)(x - y)(x^2 + xy + y)(x^2 - xy + y^2)}$.

Solution:

$$(x + y)(x - y)(x^2 + xy + y)(x^2 - xy + y^2)$$

Rearrange the terms

$$= (x - y)(x^2 + xy + y)(x + y)(x^2 - xy + y^2)$$

$$= (x^3 - y^3)(x^3 + y^3)$$

$$= (x^3)^2 - (y^3)^2 = x^6 - y^6$$

Exercise 4.2

1. Solve:

i. If $a+b=10$ and $a-b=6$, then find the value of (a^2+b^2).

Solution:

Given:

$a+b=10$ and $a-b=6$

To find the value of a^2+b^2, we use the above formula

$(a+b)^2+(a-b)^2=2(a^2+b^2)$

Now by putting the values of $a+b=10$ and $a-b=6$, we get

$(10)^2+(6)^2=2(a^2+b^2)$

$100+36=2(a^2+b^2)$

$136=2(a^2+b^2)$

$a^2+b^2=\frac{136}{2}$

$a^2+b^2=68$

ii. If $a+b=5$ and $a-b=\sqrt{17}$, then find the value of ab.

Solution:

Given:

$a+b=5$ and $a-b=\sqrt{17}$

To find the value of a^2+b^2, we use the above formula

$(a+b)^2-(a-b)^2=4ab$

Now by putting the values of $a+b=5$ and $a-b=\sqrt{17}$, we get

$(5)^2-\left(\sqrt{17}\right)^2=4ab$

$25-17=4ab$

$8=4ab$

$ab=\frac{8}{4}$

$ab=2$

2. If $a^2 + b^2 + c^2 = 45$ and $a + b + c = -1$, then find the value of $ab + bc + ca$.

Solution:

Given:

$a^2 + b^2 + c^2 = 45$ and $a + b + c = -1$

To find the value of $ab + bc + ca$, we use the formula

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

Now by putting the values of $a^2 + b^2 + c^2 = 45$ and $a + b + c = -1$, we get

$(-1)^2 = 45 + 2(ab + bc + ca)$

$1 = 45 + 2(ab + bc + ca)$

$1 - 45 = 2(ab + bc + ca)$

$-44 = 2(ab + bc + ca)$

$ab + bc + ca = \frac{-44}{2}$

$ab + bc + ca = -22$

3. If $m + n + p = 10$ and $mn + np + mp = 27$, then find the value of $m^2 + n^2 + p^2$.

Solution:

Given:

$m + n + p = 10$ and $mn + np + mp = 27$

To find the value of $m^2 + n^2 + p^2$, we use the formula

$(m + n + p)^2 = m^2 + n^2 + p^2 + 2(mn + np + mp)$

Now by putting the values of $m + n + p = 10$ and $mn + np + mp = 27$, we get

$(10)^2 = m^2 + n^2 + p^2 + 2(27)$

$100 = m^2 + n^2 + p^2 + 54$

$m^2 + n^2 + p^2 = 100 - 54$

$m^2 + n^2 + p^2 = 46$

4. If $x^2 + y^2 + z^2 = 78$ and $xy + yz + zx = 59$, then find the value of $x + y + z$.

Solution:

Given:

$x^2 + y^2 + z^2 = 78$ and $xy + yz + zx = 59$

To find the value of $x + y + z$, we use the formula

$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

Now by putting the values of $x^2 + y^2 + z^2 = 78$ and $xy + yz + zx = 59$, we get

$(x + y + z)^2 = 78 + 2(59)$

$(x + y + z)^2 = 78 + 118$

$(x + y + z)^2 = 196$

Taking square root on both sides

$\sqrt{(x + y + z)^2} = \sqrt{196}$

$x + y + z = \pm 14$

5. If $x + y + z = 12$ and $x^2 + y^2 + z^2 = 64$, then find the value of $xy + yz + zx$.

Solution:

Given:

$x^2 + y^2 + z^2 = 64$ and $x + y + z = 12$

To find the value of $xy + yz + zx$, we use the formula

$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

Now by putting the values of $x^2 + y^2 + z^2 = 64$ and $x + y + z = 12$, we get

$(12)^2 = 64 + 2(xy + yz + zx)$

$144 = 64 + 2(xy + yz + zx)$

$144 - 64 = 2(xy + yz + zx)$

$80 = 2(xy + yz + zx)$

$xy + yz + zx = \frac{80}{2}$

$xy + yz + zx = 40$

6. If $x + y = 7$ and $xy = 12$ then find the values of $x^3 + y^3$.

Solution:

Given:

$x + y = 7$ .... (1)

To find the value of $x^3 + y^3$, we use the formula

$(a + b)^3 = a^3 + 3ab(a + b) + b^3$

Taking cube on both slides of eq. (1)

$(x + y)^3 = (7)^3$

$(x)^3 + 3(x)(y)(x + y) + (y)^3 = 343$

$x^3 + 3xy(x + y) + y^3 = 343$

Now by putting the values of $x + y = 7$ and $xy = 12$ given in question

$x^3 + 3(12)(7) + y^3 = 343$

$x^3 + 252 + y^3 = 343$

$x^3 + y^3 = 343 - 252$

$x^3 + y^3 = 91$

7. If $3x + 4y = 11$ and $xy = 12$, then find the value of $27x^3 + 64y^3$.

Solution:

Given:

$3x + 4y = 11$ .... (1)

To find the value of $27x^3 + 64y^3$, we use the formula

$(a + b)^3 = a^3 + 3ab(a + b) + b^3$

Taking cube on both slides of eq. (1)

$(3x + 4y)^3 = (11)^3$

$(3x)^3 + 3(3x)(4x)(3x + 4y) + (4y)^3 = 1331$

$27x^3 + 36xy(3x + 4y) + 64y^3 = 1331$

Now putting the values of $3x + 4y = 11$ and $xy = 12$ given in question

$27x^3 + 36(12)(11) + 64y^3 = 1331$

$27x^3 + 4752 + 64y^3 = 1331$

$27x^3 + 64y^3 = 1331 - 4752$

$27x^3 + 64y^3 = -3421$

8. If $x - y = 4$ and $xy = 21$ then find the values of $x^3 - y^3$.

Solution:

Given:

$x - y = 4$ .... (1)

To find the value of $x^3 - y^3$, we use the formula

$(a - b)^3 = a^3 - 3ab(a - b) - b^3$

Taking cube on both slides of eq. (1)

$(x - y)^3 = (4)^3$

$(x)^3 - 3(x)(y)(x - y)(y)^3 = 64$

$x^3 - 3xy(x - y) - y^3 = 64$

Now putting the values of $x - y = 4$ and $xy = 21$, given in question

$x^3 - 3(21)(4) - y^3 = 64$

$x^3 - 252 - y^3 = 64$

$x^3 - y^3 = 64 + 252$

$x^3 - y^3 = 316$

9. If $5x - 6y = 13$ and $xy = 6$, then find the value of $125x^3 - 216y^3$.

Solution:

Given:

$5x - 6y = 13$ .... (1)

To find the value of $125x^3 - 216y^3$, we use the formula

$(a - b)^3 = a^3 - 3ab(a - b) - b^3$

Taking cube on both slides of eq. (1)

$(5x - 6y)^3 = (13)^3$

$(5x)^3 - 3(5x)(6y)(5x - 6y) - (6y)^3 = 2197$

$125x^3 - 90xy(5x - 6y) - 216y^3 = 2197$

Now Putting the Values of $5x - 6y = 13$ and $xy = 6$, given in question

$125x^3 - 90(6)(13) - 216y^3 = 2197$

$125x^3 - 7020 - 216y^3 = 2197$

$125x^3 - 216y^3 = 2197 + 7020$

$125x^3 - 216y^3 = 9217$

10. **If $x + \frac{1}{x} = 3$, then find the value of $x^3 + \frac{1}{x^3}$.**

Solution:

$x + \frac{1}{x} = 3$ ……………………… (1)

Taking cube on both sides of the above equation

$\left(x + \frac{1}{x}\right)^3 = (3)^3$

$(x)^3 + 3(x)\left(\frac{1}{x}\right)\left(x + \frac{1}{x}\right) + \left(\frac{1}{x}\right)^3 = 27$

$x^3 + 3\left(x + \frac{1}{x}\right) + \frac{1}{x^3} = 27$

Now put the value of $x + \frac{1}{x} = 3$, given in question

$x^3 + 3(3) + \frac{1}{x^3} = 27$

$x^3 + 9 + \frac{1}{x^3} = 27$

$x^3 + \frac{1}{x^3} = 27 - 9$

$x^3 + \frac{1}{x^3} = 18$

11. **If $x - \frac{1}{x} = 7$, then find the value of $x^3 - \frac{1}{x^3}$.**

Solution:

$$x - \frac{1}{x} = 7 \text{ .... } (1)$$

Taking cube on both sides of the above equation

$$\left(x - \frac{1}{x}\right)^3 = (7)^3$$

$$(x)^3 - 3(x)\left(\frac{1}{x}\right)\left(x - \frac{1}{x}\right) - \left(\frac{1}{x}\right)^3 = 343$$

$$x^3 - 3\left(x - \frac{1}{x}\right) - \frac{1}{x^3} = 343$$

Now put the value of $x - \frac{1}{x} = 7$, given in question

$$x^3 - 3(7) - \frac{1}{x^3} = 343$$

$$x^3 - 21 - \frac{1}{x^3} = 343$$

$$x^3 - \frac{1}{x^3} = 343 + 21$$

$$x^3 - \frac{1}{x^3} = 364$$

12. **If $3x + \frac{1}{3x} = 5$, then find the value of $27x^3 + \frac{1}{27x^3}$.**

Solution:

$$3x + \frac{1}{3x} = 5 \text{ .... } (1)$$

Taking cube on both sides of the above equation

$$\left(3x + \frac{1}{3x}\right)^3 = (5)^3$$

$$(3x)^3 + 3(3x)\left(\frac{1}{3x}\right)\left(3x + \frac{1}{3x}\right) + \left(\frac{1}{3x}\right)^3 = 125$$

$$27x^3 + 3\left(3x + \frac{1}{3x}\right) + \frac{1}{27x^3} = 125$$

Now put the value of $3x + \frac{1}{3x} = 5$, given in question

$$27x^3 + 3(5) + \frac{1}{27x^3} = 125$$

$$27x^3 + 15 + \frac{1}{27x^3} = 125$$

$$27x^3 + \frac{1}{27x^3} = 125 - 15$$

$$27x^3 + \frac{1}{27x^3} = 110$$

13. If $5x - \frac{1}{5x} = 6$, then find the value of $125x^3 - \frac{1}{125x^3}$.

Solution:

$5x - \frac{1}{5x} = 6$............................. (1)

Taking cube on both sides of the above equation

$$\left(5x - \frac{1}{5x}\right)^3 = (6)^3$$

$$(5x)^3 - 3(5x)\left(\frac{1}{5x}\right)\left(5x - \frac{1}{5x}\right) - \left(\frac{1}{5x}\right)^3 = 216$$

$$125x^3 - 3\left(5x - \frac{1}{5x}\right) - \frac{1}{125x^3} = 216$$

Now put the value of $5x - \frac{1}{5x} = 6$, given in question

$$125x^3 - 3(6) - \frac{1}{125x^3} = 216$$

$$125x^3 - 18 - \frac{1}{125x^3} = 216$$

$$125x^3 - \frac{1}{125x^3} = 216 + 18$$

$$125x^3 - \frac{1}{125x^3} = 234$$

14. Factorize:

i. $\boldsymbol{x^3 - y^3 - x + y}$

Solution:

$x^3 - y^3 - x + y$

$= (x^3 - y^3) - (x - y)$

$= (x - y)(x^2 + xy + y^2) - (x - y)$ $\quad \because x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$= (x - y)[(x^2 + xy + y^2) - 1]$ $\quad$ Taking '$(x - y)$' as a Common

$= (x - y)(x^2 + xy + y^2 - 1)$

ii. $\boldsymbol{8x^3 - \frac{1}{27y^3}}$

Solution:

$8x^3 - \dfrac{1}{27y^3}$

$= (2x)^3 - \left(\dfrac{1}{3y}\right)^3$

$= \left(2x - \dfrac{1}{3y}\right)\left[(2x)^2 + (2x)\left(\dfrac{1}{3y}\right) + \left(\dfrac{1}{3y}\right)^2\right]$

$= \left(2x - \dfrac{1}{3y}\right)\left(4x^2 + \dfrac{2x}{3y} + \dfrac{1}{9y^2}\right)$

15. Find the products, using formulas.

i. $\boldsymbol{(x^2 + y^2)(x^4 - x^2y^2 + y^4)}$

Solution:

$(x^2 + y^2)(x^4 - x^2y^2 + y^4)$

$= (x^2 + y^2)[(x^2)^2 - (x^2)(y^2) + (y^2)^2]$

$= (x^2)^3 + (x^2)^3$

$= x^6 + y^6$

ii. $(\boldsymbol{x^3 - y^3})(\boldsymbol{x^6 + x^3y^3 + y^6})$

Solution:

$(x^3 - y^3)(x^6 + x^3y^3 + y^6)$

$= (x^3 - y^3)[(x^3)^2 + (x^3)(y^3) + (y^3)^2]$

$= (x^3)^3 - (y^3)^3$

$= x^9 - y^9$

iii. $(\boldsymbol{x - y})(\boldsymbol{x + y})(\boldsymbol{x^2 + y^2})(\boldsymbol{x^2 + xy + y^2})(\boldsymbol{x^2 - xy + y^2})(\boldsymbol{x^4 - x^2y^2 + y^4})$

Solution:

$(x - y)(x + y)(x^2 + y^2)(x^2 + xy + y^2)(x^2 - xy + y^2)(x^4 - x^2y^2 + y^4)$

$= [(x - y)(x^2 + xy + y^2)][(x + y)(x^2 - xy + y^2)][(x^2 + y^2)(x^4 - x^2y^2 + y^4)]$

$= (x^3 - y^3)(x^3 + y^3)(x^2 + y^2)[(x^2)^2 - (x^2)(y^2) + (y^2)^2]$

$= (x^3 - y^3)(x^3 + y^3)(x^2)^3 + (x^2)^3$

$= (x^3 - y^3)(x^3 + y^3)(x^6 + y^6)$

$= [(x^3)^2 - (y^3)^2](x^6 + y^6)$ $\quad \because a^2 - b^2 = (a + b)(a - b)$

$= (x^6 - y^6)(x^6 + y^6)$

$= (x^6)^2 - (y^6)^2$

$= x^{12} - y^{12}$

iv. $(\boldsymbol{2x^2 - 1})(\boldsymbol{2x^2 + 1})(\boldsymbol{4x^4 + 2x^2 + 1})(\boldsymbol{4x^4 - 2x^2 + 1})$

Solution:

$(2x^2 - 1)(2x^2 + 1)(4x^4 + 2x^2 + 1)(4x^4 - 2x^2 + 1)$

$= [(2x^2 - 1)(4x^4 + 2x^2 + 1)][(2x^2 + 1)(4x^4 - 2x^2 + 1)]$

$= [(2x^2)^3 - (1)^3][(2x^2)^3 + (1)^3]$

$= (8x^6 - 1)(8x^6 + 1)$

$= (8x^6)^2 - (1)^2 = 64x^{12} - 1$

Surds:

An irrational radical with rational radicand is called surd.

Hence the radical $\sqrt[n]{a}$ is a surd if:

i. a is rational.
ii. the result $\sqrt[n]{a}$ is irrational.

For Example:

$\sqrt{3}, \sqrt{\frac{2}{3}}, \sqrt[4]{10}$ are surds.

But $\sqrt{\pi}$ and $\sqrt{2+\sqrt{5}}$ are not surds because π and $2+\sqrt{5}$ are not rational.

Operation on Surds:

i. Addition and Subtraction of Surds:

Similar surds can be added or subtracted into single term.

Example: Simplify by combining similar terms of $4\sqrt{3} - 3\sqrt{27} + 2\sqrt{75}$.

Solution:

$$4\sqrt{3} - 3\sqrt{27} + 2\sqrt{75}$$

$$= 4\sqrt{3} - 3\sqrt{9 \times 3} + 2\sqrt{25 \times 3}$$

$$= 4\sqrt{3} - 3\sqrt{9} \times \sqrt{3} + 2\sqrt{25} \times \sqrt{3}$$

$$= 4\sqrt{3} - 3\sqrt{(3)^2} \times \sqrt{3} + 2\sqrt{(5)^2} \times \sqrt{3}$$

$$= 4\sqrt{3} - 3(3) \times \sqrt{3} + 2(5) \times \sqrt{3}$$

$$= 4\sqrt{3} - 9\sqrt{3} + 10\sqrt{3}$$

$$= (4 - 9 + 10)\sqrt{3}$$

$$= 5\sqrt{3}$$

ii. Multiplication of Surds:

We can multiply surds of the same order by making the use of the following law of surds.

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$$

Example: Simplify and express the answer in the simplest form of $\sqrt{14}\sqrt{35}$.

Solution:

$\sqrt{14}\sqrt{35}$

$= \sqrt{14 \times 35}$

$= \sqrt{7 \times 2 \times 7 \times 5}$

$= \sqrt{(7)^2 \times 2 \times 5}$

$= \sqrt{(7)^2} \times \sqrt{10}$

$= 7\sqrt{10}$

iii. Division of Surds:

We can divide surds of the same order by making the use of the following law of surds.

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Exercise 4.3

1. Express each of the following surd in the simplest form.

i. $\sqrt{180}$

Solution:

$\sqrt{180}$

$= \sqrt{2 \times 2 \times 3 \times 3 \times 5}$

$= \sqrt{2^2 \times 3^2 \times 5}$

$= \sqrt{2^2} \times \sqrt{3^2} \times \sqrt{5}$

$= 2 \times 3\sqrt{5}$

$= 6\sqrt{5}$

2	180
2	90
3	45
3	15
5	5
	1

ii. $3\sqrt{162}$

Solution:

$3\sqrt{162}$

$= 3\sqrt{2 \times 3 \times 3 \times 3 \times 3}$

$= 3\sqrt{2 \times 3^2 \times 3^2}$

$= 3\sqrt{2} \times \sqrt{3^2} \times \sqrt{3^2}$

$= 3\sqrt{2} \times 3 \times 3$

$= 27\sqrt{2}$

iii. $\mathbf{\frac{3}{4}\sqrt[3]{128}}$

Solution:

$\frac{3}{4}\sqrt[3]{128}$

$= \frac{3}{4}\sqrt[3]{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}$

$= \frac{3}{4}\sqrt[3]{2^3 \times 2^3 \times 2}$

$= \frac{3}{4}\sqrt[3]{2^3} \times \sqrt[3]{2^3} \times \sqrt[3]{2}$

$= \frac{3}{4} \times 2 \times 2 \times \sqrt[3]{2}$

$= \frac{3}{4} \times 4 \times \sqrt[3]{2}$

$= 3\sqrt[3]{2}$

iv. $\mathbf{\sqrt[5]{96x^6y^7z^8}}$

Solution:

$\sqrt[5]{96x^6y^7z^8}$

$= \sqrt[5]{2 \times 2 \times 2 \times 2 \times 2 \times 3x^6y^7z^8}$

$= \sqrt[5]{2^5 \times 3x^6y^7z^8}$

$= \sqrt[5]{2^5} \times \sqrt[5]{3} \times \sqrt[5]{x^6} \times \sqrt[5]{y^7} \times \sqrt[5]{z^8}$

$= 2 \times 3^{\frac{1}{5}} \times x^{\frac{6}{5}} \times y^{\frac{7}{5}} \times z^{\frac{8}{5}}$

$= 2xyz(xy^2z^3)^{\frac{1}{5}}$

$= 2xyz\sqrt[5]{xy^2z^3}$