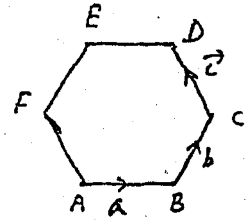
Exercise # 3.1

Q:1 ABCDEF is a regular hexagon. $\overrightarrow{AB} = a$, $\overrightarrow{BC} = b$ and $\overrightarrow{CD} = c$. State the following vectors as scalar multiple of a, b or c.



Sol

(a) $\overrightarrow{DE} = -\overrightarrow{AB} = -a$

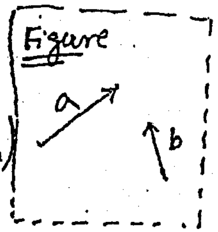
(b) $\overrightarrow{EF} = -\overrightarrow{BC} = -b$

(c) $\overrightarrow{FA} = -\overrightarrow{CB} = -c$

(d) $\overrightarrow{AD} = 2b$ because in a regular hexagon the diagnol parallel to any side is double of it.

(e) $\overrightarrow{BE} = 2c$

Q:2 Given the vectors a and b as shown in the figure, draw the vectors:



(a) $a + 2b$

Sol First multiply b by 2 (i:e Double the magnitude of a) and then add with a by head to tail rule

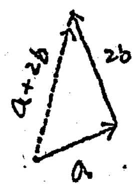


figure for (a)

(b) $2a - b$

Sol First multiply a by 2 (Double its magnitude) and the add (-b) to it. (−b means reverse the direction of b)

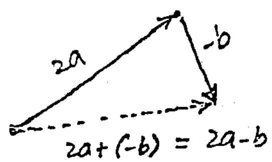


Figure for (b)

(c) $3a - 2b$

Sol First multiply a by 3 (Magnitude of a becomes 3 times)
then " b by 2
and then subtract by head to tail rule

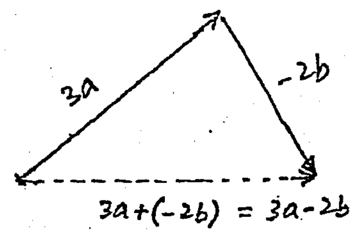


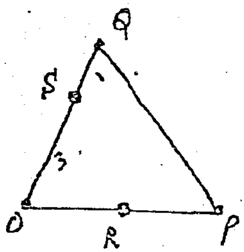
Figure for (c)

Q:3 In ΔOPQ, $\overrightarrow{OP} = p$, $\overrightarrow{OQ} = q$, R is the mid point of $\overrightarrow{OP}$ and S lies on $\overrightarrow{OQ}$ such that $|OS| = 3|SQ|$. State in terms of p and q.

Sol

(a) $\overrightarrow{OR} = \dfrac{\overrightarrow{OP}}{2} = \dfrac{p}{2}$

(b) $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$
$= -p + q$
$= q - p$

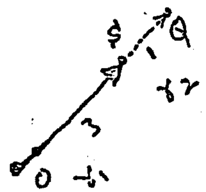


mid point of $\overrightarrow{OP}$

$\overrightarrow{OP} = p$
$\overrightarrow{OQ} = q$

(c) OS'

Since $\overrightarrow{OS'} : \overrightarrow{S'Q} = 3:1$
$\qquad\qquad r_1 \quad r_2$



By formula of finding the position vector of an internal point

$$\overrightarrow{OS'} = \frac{r_1 q + r_2(0)}{r_1+r_2}$$

$\because$ Position vector of origin $= 0$

$$\overrightarrow{OS'} = \frac{3q+0}{3+1} \Rightarrow \boxed{\overrightarrow{OS'} = \frac{3}{4}q}$$

(d) $\overrightarrow{RS'}$

Sol

$$\begin{aligned}
\overrightarrow{RS'} &= \overrightarrow{RO} + \overrightarrow{OS'} \\
&= -\overrightarrow{OR} + \overrightarrow{OS'} \\
&= -\frac{p}{2} + \frac{3}{4}q \\
&= \frac{-2p+3q}{4} \\
&= \frac{3q-2p}{4}
\end{aligned}$$

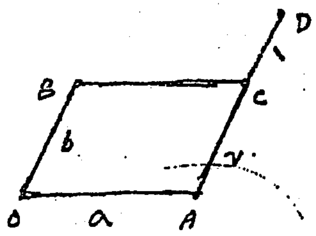
$\overrightarrow{RS'} = \frac{3}{4}q - \frac{p}{2}$ Ans

Q.4: OACB is a parallelogram with $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$, $\overrightarrow{AC}$ is extended to D where $|AC| = 2|CD|$. Find in terms of a and b.

Sol.

(a) $\overrightarrow{AD}$

$$\begin{aligned}
&= \overrightarrow{AC} + \overrightarrow{CD} \\
&= \overrightarrow{AC} + \frac{\overrightarrow{AC}}{2} \\
&= \frac{3}{2}\overrightarrow{AC} \\
&= \frac{3}{2}b \quad \because AC = OB = b
\end{aligned}$$



(b) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

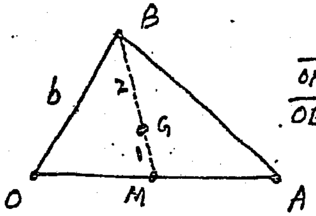
$$\begin{aligned}
&= \overrightarrow{OA} + \overrightarrow{AC} + \overrightarrow{CD} \\
&= \overrightarrow{OA} + \overrightarrow{AC} + \frac{\overrightarrow{AC}}{2} \\
&= \overrightarrow{OA} + \frac{3}{2}\overrightarrow{AC} \\
&= a + \frac{3}{2}b \quad \text{Ans}
\end{aligned}$$

(c) $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

$$\begin{aligned}
&= a + \frac{\overrightarrow{AC}}{2} \\
&= a + \frac{b}{2} \quad \text{Ans}
\end{aligned}$$

Q:5 OAB is a triangle with $\overrightarrow{OA} = a$, $\overrightarrow{OB} = b$. M is the mid point of OA and G lies on MB such that $|MG| = \frac{1}{2}|GB|$. State in terms of a and b

Sol



$\overrightarrow{OA} = a$
$\overrightarrow{OB} = b$

(a) $\overrightarrow{OM} = \frac{\overrightarrow{OA}}{2} = \frac{a}{2}$

(b) $\overrightarrow{MB} = \overrightarrow{MO} + \overrightarrow{OB}$ (H to T rule)

$= -\frac{a}{2} + b$

$= b - \frac{a}{2}$

(c) $\overrightarrow{MG} = \overrightarrow{MO} + \overrightarrow{OB} + \overrightarrow{BG}$ (Head to tail rule)

$= -\overrightarrow{OM} + \overrightarrow{OB} - \overrightarrow{GB}$

$\overrightarrow{MG} = -\frac{a}{2} + b - 2MG \quad \because GB = -2MG$

$\overrightarrow{MG} + 2\overrightarrow{MG} = -\frac{a}{2} + b$

$3\overrightarrow{MG} = \frac{-a}{2} + b$

$\overrightarrow{MG} = \frac{-a}{6} + \frac{b}{3}$ Ans

(d) $\overrightarrow{OG} = \overrightarrow{OM} + \overrightarrow{MG}$

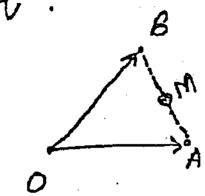
$= \frac{a}{2} + \left(\frac{-a}{6} + \frac{b}{3}\right)$

$= \frac{a}{2} - \frac{a}{6} + \frac{b}{3}$

$= \frac{3a - a + 2b}{6} = \frac{2a + 2b}{6} = \frac{a+b}{3}$ Ans

Q:6 $\overrightarrow{OA} = p + q$, $\overrightarrow{OB} = 2p - q$ where p and q are two vectors and M is the mid point of AB. Find in terms of p and q. CH-03 P-02

Sol



(a) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$= -(p+q) + (2p-q)$

$= -p - q + 2p - q$

$= p - 2q$

(b) $\overrightarrow{AM} = \frac{\overrightarrow{AB}}{2} = \frac{-\overrightarrow{OA} + \overrightarrow{OB}}{2} = \frac{-(p+q) + (2p-q)}{2} = \frac{-p-q+2p-q}{2}$

$= \frac{p-2q}{2}$ Ans

(c) $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= (p+q) + \left(\frac{p-2q}{2}\right)$

$= \frac{2p + 2q + p - 2q}{2} = \frac{3p}{2}$ Ans

Q:7 Given that $p = 3a - b$ and $q = 2a - 3b$. Find numbers x and y such that $xp + yq = a + 9b$

Sol

$xp + yq = a + 9b$

$\Rightarrow x(3a - b) + y(2a - 3b) = a + 9b$

$\Rightarrow 3ax - bx + 2ay - 3by = a + 9b$

$\Rightarrow (3x + 2y)a + (-x - 3y)b = 1a + 9b$

Compare the coefficients of a and b, we get

$\Rightarrow \quad 3x + 2y = 1 \longrightarrow (i)$

$\quad -x - 3y = 9 \longrightarrow (ii)$

xing equ (ii) by 3 and then add with equ (i)

$$\begin{aligned} 3x + 2y &= 1 \\ -3x - 9y &= 27 \\ \hline -7y &= 28 \Rightarrow \boxed{y = -4} \end{aligned}$$

Now equ (ii) $\quad -x - 3y = 9$

$\Rightarrow -x - 3(-4) = 9$

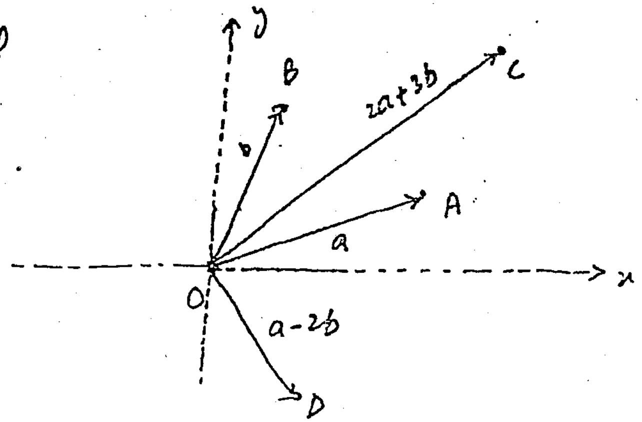
$\Rightarrow -x + 12 = 9$

$\Rightarrow -x = -3 \Rightarrow \boxed{x = 3}$

Hence $\left.\begin{aligned} x &= 3 \\ y &= -4 \end{aligned}\right\}$ Ans

<u>Q:8</u> The position vectors of four points A, B, C, D are a, b; $2a+3b$ and $a-2b$ respectively. Find $\overrightarrow{AC}$, $\overrightarrow{DB}$, $\overrightarrow{BC}$ and $\overrightarrow{CD}$ in terms of a and b.

<u>Sol</u>



<u>Sol</u>

(i) $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= -\overrightarrow{OA} + \overrightarrow{OC}$

$= -a + (2a + 3b)$

$= a + 3b$ Ans

(ii) $\overrightarrow{DB} = \overrightarrow{DO} + \overrightarrow{OB}$

$= -\overrightarrow{OD} + \overrightarrow{OB}$

$= -(a-2b) + b$

$= -a + 2b + b$

$= -a + 3b$ Ans

(iii) $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$= -\overrightarrow{OB} + \overrightarrow{OC}$

$= -(b) + (2a+3b)$

$= 2a + 2b$ Ans

(iv) $\overrightarrow{CD} = \overrightarrow{CO} + \overrightarrow{OD}$

$= -\overrightarrow{OC} + \overrightarrow{OD}$

$= -(2a+3b) + (a-2b)$

$= -2a - 3b + a - 2b$

$= -a - 5b$

<u>Golden words</u>

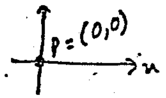
Try not to become a man of success but rather to become a man of value.

"Albert Einstein"

Exercise # 3.3

Q:1 Find the position vectors of the following points.

(i) $P = (0,0)$



Sol: Position vector $= \overrightarrow{OP}$

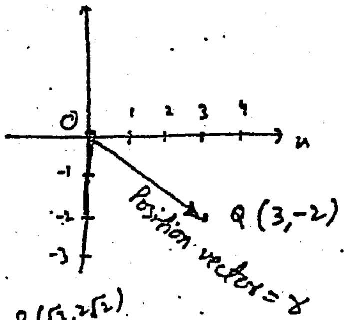
$r = P - O$

$= (0,0) - (0,0)$

$r = (0,0)$

$r = 0i + 0j$

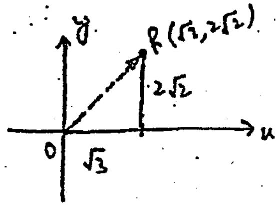
(ii) $Q = (3,-2)$



Sol: $r = \overrightarrow{OQ}$

$= 3i - 2j$

(iii) $R = (\sqrt{3}, 2\sqrt{2})$

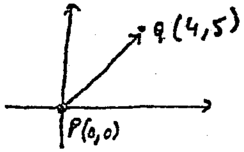


Sol: $r = \overrightarrow{OR}$

$= \sqrt{3}\,i + 2\sqrt{2}\,i$

Q:2 Express the vector $\overrightarrow{PQ}$ in the form $xi + yj$

(i) $P(0,0)$, $Q(4,5)$



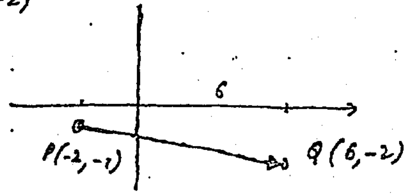
Sol: $\overrightarrow{PQ} = \vec{Q} - \vec{P}$

$= (4,5) - (0,0)$

$= (4i + 5j) - (0i + 0j)$

$= 4i + 5j$ Ans

(ii) $P = (-2,-1)$ $Q = (6,-2)$



Sol: $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

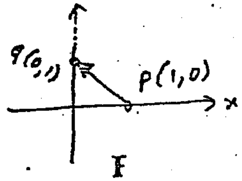
$= (6,-2) - (-2,-1)$

$= (6i - 2j) - (-2i - 1j)$

$= 6i - 2j + 2i + 1j$

$= 8i - 1j$ Ans

(iii) $P = (1,0)$ $Q = (0,1)$



Sol: $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (0,1) - (1,0)$

$= (0i + 1j) - (1i + 0j)$

$= 0i + 1j - 1i - 0j$

$= -1i + 1j$ Ans

Q:3 If $a = 3i - 5j$ and $b = -2i + 3j$, then find

(i) $a + 2b$

Sol: $a + 2b = (3i - 5j) + 2(-2i + 3j)$

$= 3i - 5j - 4i + 6j$

$= -i + j$ Ans

(ii) $3a - 2b$

Sol: $3a - 2b = 3(3i - 5j) - 2(-2i + 3j)$

$= 9i - 15j + 4i - 6j$

$= 13i - 21j$ Ans

(iii) $2(a - b)$

Sol: $2(a - b) = 2\{(3i - 5j) - (-2i + 3j)\}$

$= 2\{3i - 5j + 2i - 3j\} = 2\{5i - 8j\} = 10i - 16j$ Ans

(iv) $|a+b|$

Sol $a+b = (3i-5j) + (-2i+3j)$

$\Rightarrow a+b = i-2j$

take magnitude, we get

$\Rightarrow |a+b| = \sqrt{1^2+(-2)^2} = \sqrt{1+4} = \sqrt{5}$

(v) $|a| - |b|$

Sol $|a| - |b| = |3i-5j| - |-2i+3j|$

$= \sqrt{3^2+(-5)^2} - \sqrt{(-2)^2+3^2}$

$= \sqrt{9+25} - \sqrt{4+9}$

$= \sqrt{34} - \sqrt{13}$ Ans

(vi) $\dfrac{|a|}{|b|}$

Sol $\dfrac{|a|}{|b|} = \dfrac{|3i-5j|}{|-2i+3j|} = \dfrac{\sqrt{3^2+(-5)^2}}{\sqrt{(-2)^2+3^2}} = \dfrac{\sqrt{9+25}}{\sqrt{4+9}} = \dfrac{\sqrt{34}}{\sqrt{13}}$

$= \sqrt{\dfrac{34}{13}}$ Ans

Q:4 Find the unit vector having the same direction as the vectors given below.

(i) $3i$

Sol Let $\vec{v} = 3i$

$\Rightarrow |v| = \sqrt{3^2} = 3$

then $\hat{v} = \dfrac{\vec{v}}{|v|} = \dfrac{3i}{3} = i$

Hence $\hat{v} = i$

Note
A unit vector has the same direction as the given vector and $\hat{v} = \dfrac{\vec{v}}{|v|}$

(ii) $i-j$

Sol Let $\vec{v} = i-j$

$\Rightarrow |v| = \sqrt{1^2+(-1)^2} = \sqrt{2}$

Then $\hat{v} = \dfrac{\vec{v}}{|v|} = \dfrac{i-j}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}i - \dfrac{1}{\sqrt{2}}j$ Ans

(iii) $3i-4j$

Sol Let $\vec{v} = 3i-4j$

$\Rightarrow |v| = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

Then $\hat{v} = \dfrac{\vec{v}}{|v|} = \dfrac{3i-4j}{5} = \dfrac{3}{5}i - \dfrac{4}{5}j$ Ans

(iv) $\dfrac{\sqrt{3}}{2}i - \dfrac{1}{2}j$

Sol Let $\vec{v} = \dfrac{\sqrt{3}}{2}i - \dfrac{1}{2}j$

$\Rightarrow |v| = \sqrt{\left(\dfrac{\sqrt{3}}{2}\right)^2 + \left(\dfrac{-1}{2}\right)^2} = \sqrt{\dfrac{3}{4}+\dfrac{1}{4}} = \sqrt{\dfrac{4}{4}} = 1$

Then $\hat{v} = \dfrac{\vec{v}}{|v|} = \dfrac{\frac{\sqrt{3}}{2}i - \frac{1}{2}j}{1} = \dfrac{\sqrt{3}}{2}i - \dfrac{1}{2}j$ Ans

Q:5 If $r = i-9j$, $a = i+2j$, $b = 5i-j$. Determine the real #s p and q such that $r = pa+qb$

Sol As $r = pa + qb$

$\Rightarrow i-9j = p(i+2j) + q(5i-j)$

$\Rightarrow i-9j = pi + 2pj + 5qi - qj$

$\Rightarrow 1i - 9j = (p+5q)i + (2p-q)j$

compare b.s, we get

$1 = p + 5q \longrightarrow (i)$
$-9 = 2p - q \longrightarrow (ii)$

Eqn (i) multiplied by 2 and then subtract eqn (ii), we get

$$\begin{array}{l} 2 = 2p + 10q \\ \underset{+}{-}9 = \underset{-}{2}p \underset{+}{-} q \\ \hline 11 = 11q \end{array} \Rightarrow \boxed{1 = q}$$

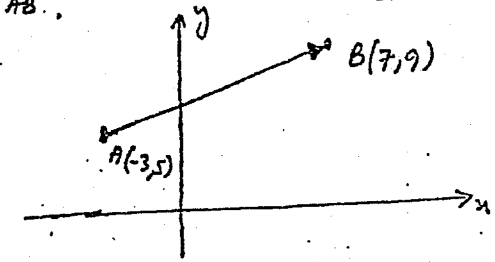
Eqn(i) $\Rightarrow$ $1 = p + 5q$

$1 = p + 5(1) \Rightarrow 1 = p + 5 \Rightarrow \boxed{-4 = p}$

Hence $\boxed{q = 1 \ \& \ p = -4}$ Ans

Q:6 Find the length of the vector $\overrightarrow{AB}$ from the point $A(-3,5)$ to $B(7,9)$. Also find a unit vector in the direction of $\overrightarrow{AB}$.

Sol



$\overrightarrow{AB} = \text{Head} - \text{Tail}$
$= (7,9) - (-3,5)$
$= (10,4)$

$\overrightarrow{AB} = 10i + 4j$

Then length will be $|AB| = \sqrt{10^2 + 4^2} = \sqrt{116} = \sqrt{4 \times 29} = \sqrt{4}\sqrt{29}$

$\Rightarrow |AB| = 2\sqrt{29}$

Now Unit vector of AB is

$$\hat{AB} = \frac{\overrightarrow{AB}}{|AB|} = \frac{10i + 4j}{2\sqrt{29}} = \frac{10i}{2\sqrt{29}} + \frac{4}{2\sqrt{29}}j = \frac{5i}{\sqrt{29}} + \frac{2}{\sqrt{29}}j$$

Q:7 If $p = 2i + 3j$ and $q = i - j$ then find numbers x and y such that $xp + yq = -4i - 11j$.

Sol

$xp + yq = -4i - 11j$

Put the values of p and q

$\Rightarrow x(2i + 3j) + y(i - j) = -4i - 11j$

$\Rightarrow 2xi + 3xj + yi - yj = -4i - 11j$

$\Rightarrow (2x + y)i + (3x - y)j = -4i - 11j$

compare b.s, we get

$2x + y = -4 \longrightarrow (i)$
$3x - y = -11 \longrightarrow (ii)$

Eqn(i) + Eqn (ii), we get

$5x = -15 \Longrightarrow \boxed{x = -3}$ Ans

Now eqn (i) $2x + y = -4$
$2(-3) + y = -4$
$-6 + y = -4 \Rightarrow \boxed{y = 2}$ Ans

Q:8 If $p = 2i - j$ and $q = xi + 3j$. Then find x such that $|p + q| = 5$.

Sol

$p + q = (2i - j) + (xi + 3j)$

$\Rightarrow p + q = (2 + x)i + (3 - 1)j$

$\Rightarrow p + q = (2 + x)i + 2j$

take magnitude

$|p + q| = \sqrt{(2 + x)^2 + 2^2}$

$\Rightarrow 5 = \sqrt{4 + x^2 + 4x + 4}$

$\Rightarrow \quad 5 = \sqrt{x^2+4x+8}$

squaring b.s

$25 = x^2+4x+8$

$\Rightarrow \quad 1x^2+4x-17=0$

By quadratic formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{(4)^2-4(1)(-17)}}{2(1)}$$

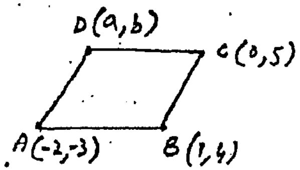
$$x = \frac{-4 \pm \sqrt{16+68}}{2} = \frac{-4 \pm \sqrt{84}}{2} = \frac{-4 \pm \sqrt{4 \times 21}}{2}$$

$$\Rightarrow x = \frac{-4 \pm 2\sqrt{21}}{2} \Rightarrow x = -2 \pm \sqrt{21}$$

Hence $x = -2 \pm \sqrt{21}$ Ans

Q:9 If ABCD is a parallelogram such that the coordinates of the vertices A, B and C are respectively $(-2,-3)$, $(1,4)$ and $(0,5)$. Find the coordinates of the vertex D.

Sol Let $D = (a,b)$



Now $\overrightarrow{AD} = \overrightarrow{BC}$

$\Rightarrow (a-(-2))i + (b-(-3))j = (0-1)i + (5-4)j$

$\Rightarrow (a+2)i + (b+3)j = -1i + 1j$

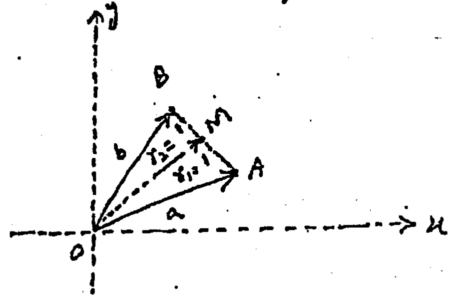
compare the coefficients of i and j, we get

$a+2=-1$ & $b+3=1$

$\Rightarrow \boxed{a=-3}$ Ans $\boxed{b=-2}$ Ans

Q:10 If a and b are position vectors of points A and B respectively, then prove that the position vector of the mid point of the line segment joining A and B is $\frac{a+b}{2}$.

Sol In the figure



$\overrightarrow{OA} = a$

$\overrightarrow{OB} = b$

M is the mid point of AB

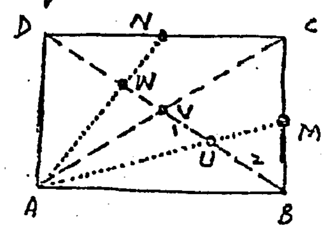
Then $\overrightarrow{OM} = \frac{a(r_2)+b(r_1)}{r_1+r_2}$

$\overrightarrow{OM} = \frac{a(1)+b(1)}{1+1}$

$\Rightarrow \boxed{\overrightarrow{OM} = \frac{a+b}{2}}$ Hence proved

Q:11 Using vectors, prove that the line that passes through the mid points of adjacent sides of a rectangle divides one of the diagnols in the ratio 1:3.

Sol Let M and N are the mid points of adjacent sides BC and CD as shown in the figure.



AC and BD are the diagnols

Now $\overline{BU} = \frac{2}{3}\overline{BV}$ $\because$ Medians intersect each other in 2:1

$\Rightarrow \overline{BU} = \frac{2}{3}\left\{\frac{\overline{BD}}{2}\right\}$ $\because 2BV = BD$

$\Rightarrow \overrightarrow{BU} = \frac{1}{3}\overrightarrow{BD}$ or $3\overrightarrow{BU} = \overrightarrow{BD}$ $\Rightarrow \overrightarrow{BU} : \overrightarrow{BD} = 1:3$

Now $\overrightarrow{DW} = \frac{2}{3}\overrightarrow{DV}$

$\Rightarrow \overrightarrow{BW} = \frac{2}{3}\left(\frac{DB}{2}\right)$

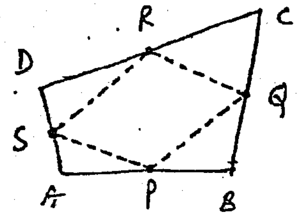
$\Rightarrow \overrightarrow{BW} = \frac{\overrightarrow{DB}}{3} \Rightarrow 3\overrightarrow{BW} = \overrightarrow{DB}$ Hence $\boxed{\overrightarrow{BW} : \overrightarrow{DB} = 1:3}$

Hence $\overrightarrow{AM}$ and $\overrightarrow{AN}$ divides the diagnols $\overline{BD}$ and 1:3

Q12 Prove that the line segments joining the mid points of consecutive sides of a quadrilateral determine a parallelogram.

Sol: Let the position vectors of A, B, C & D with respect to some origin are (i:e the vector OA, OB, OC & $\overline{OD}$ are) a, b, c & d respectively.

Let P, Q, R & S are the mid points of the quadrilateral ABCD.



Then $\overrightarrow{OP} = \frac{a+b}{2}$ & $\overrightarrow{OQ} = \frac{b+c}{2}$

Therefore $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \frac{b+c}{2} - \frac{a+b}{2}$

$= \frac{b+c-a-b}{2} = \frac{c-a}{2}$

Now $\overrightarrow{OR} = \frac{c+d}{2}$ and $\overrightarrow{OS} = \frac{a+d}{2}$

Therefore $\overrightarrow{SR} = \overrightarrow{OR} - \overrightarrow{OS} = \frac{c+d}{2} - \frac{a+d}{2} = \frac{c+d-a-d}{2} = \frac{c-a}{2}$

Hence $\overrightarrow{PQ} = \overrightarrow{SR} \longrightarrow (i)$

Now $\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP} = \frac{a+d}{2} - \frac{a+b}{2} = \frac{a+d-a-b}{2} = \frac{d-b}{2}$

& $\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \frac{c+d}{2} - \frac{b+c}{2} = \frac{c+d-b-c}{2} = \frac{d-b}{2}$

Hence $\overrightarrow{PS} = \overrightarrow{QR} \longrightarrow (ii)$

From eqns (i), (ii), (iii), (iv) it is proved that PSQR form a parallelogram.

Exercise # 3.3

Q:1 Find the components of the vector $\overrightarrow{P_1P_2}$

(i) $P_1 = (5, -2, 1)$, $P_2 = (2, 4, 2)$

$\overrightarrow{P_1P_2} = (2-5, 4-(-2), 2-1)$ i:e Head - Tail

$\overrightarrow{P_1P_2} = (-3, 6, 1)$

$\Rightarrow \overrightarrow{P_1P_2} = -3i + 6j + 1k$

Hence the components are $-3, 6, 1$ respectively.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; x \;\; y \;\; z$

(ii) $P_1 = (0,0,0)$ $P_2 = (-2, 5, 1)$.

$\overrightarrow{P_1P_2} = (-2-0, 5-0, 1-0)$

$= (-2, 5, 1) = -2i + 5j + 1k$

Hence components are $-2, 5$ & 1.

(iii) $P_1 = (2, 1, -3)$ $P_2 = (7, 1, -3)$

$\overrightarrow{P_1P_2} = (7-2, 1-1, -3-(-3))$

$= (5, 0, 0)$

$= 5i + 0j + 0k$

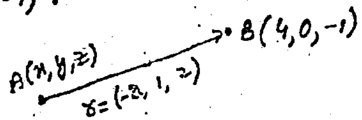
Hence components are $5, 0, 0$.

Q:2 Find the initial point of the vector $\gamma = (-2, 1, 2)$ and the terminal point is $(4, 0, -1)$.

Sol: Terminal point $= B = (4, 0, -1)$

Initial point $= A = ?$

Let $A = (x, y, z)$



Now Head − Tail = r

$\Rightarrow (4,0,-1) - (x,y,z) = (-2,1,2)$

$\Rightarrow (4-x,\ 0-y,\ -1-z) = (-2,1,2)$

compare b.s, we get

$4-x=-2$, $0-y=1$, $-1-z=2$

$\boxed{6=x}$, $\boxed{y=-1}$, $\boxed{-3=z}$

Hence $A=(6,-1,-3)$ Ans

Q:3: Find the terminal point of the vector $r=i+3j-3k$ if the initial point is $(-2,1,4)$

Sol $r=(1,3,-3)$

Initial point $= A = (-2,1,4)$

Terminal point $= B = (x,y,z)$

$A(-2,1,4) \xrightarrow{r=(1,3,-3)} B(x,y,z)$

Now (Head − Tail) $= r$

$\Rightarrow (x,y,z)-(-2,1,4) = (1,3,-3)$

$\Rightarrow (x+2,\ y-1,\ z-4) = (1,3,-3)$

compare b.s, we get

$x+2=1$, $y-1=3$, $z-4=-3$

$\Rightarrow \boxed{x=-1}$ $\boxed{y=4}$ $\boxed{z=1}$

Hence $B=(-1,4,1)$ Ans

Q:4 Let $u=i+2j-3k$, $v=2i-j+2k$, $w=3i-j+5k$

Find

(i) $u-2v = (i+2j-3k)-2(2i-j+2k)$

$= i+2j-3k-4i+2j-4k$

$\Rightarrow u-2v = -3i+4j-7k$ Ans

(ii) $3v+2w = 3(2i-j+2k)+2(3i-j+5k)$

$= 6i-3j+6k + 6i-2j+10k$

$= 12i-5j+16k$ Ans

(iii) $3u-(2v+w)$

$= 3(i+2j-3k) - \{2(2i-j+2k)+3i-j+5k\}$

$= (3i+6j-9k) - (4i-2j+4k+3i-j+5k)$

$= (3i+6j-9k) - (7i-3j+9k)$

$= 3i+6j-9k-7i+3j-9k$

$= -4i+9j-18k$ Ans

Q:5 $p = i-3j+2k$

$q = i+j$ and $r = 2i+2j-4k$. Find

(i) $|p+q-r|$

Sol $p+q-r = (i-3j+2k)+(i+j)-(2i+2j-4k)$

$p+q-r = (2i-2j+2k)-(2i+2j-4k)$

$p+q-r = 2i-2j+2k-2i-2j+4k$

$p+q-r = 0i-4j+6k$

Then $|p+q-r| = |0i-4j+6k|$

$= \sqrt{0^2+(-4)^2+6^2}$

$= \sqrt{0+16+36} = \sqrt{52}$ Ans

(ii) $|p|+|q|$

Sol $|p| = \sqrt{1^2+(-3)^2+2^2} = \sqrt{1+9+4} = \sqrt{14}$

$|q| = \sqrt{1^2+1^2} = \sqrt{2}$

Then $|p|+|q| = \sqrt{14}+\sqrt{2}$ Ans

For Q:6 — Q:12

$A = i+j+k$, $B = i-j+2k$, $C = j+k$, $D = 2i+j$

Q:6 Find $|AB|$ & $|BD|$

Sol

$\vec{AB} = \vec{OB} - \vec{OA}$

$= \vec{B} - \vec{A} = (i-j+2k) - (i+j+k)$

$= i-j+2k-i-j-k$

$\Rightarrow \vec{AB} = -2j+k$

$\Rightarrow |AB| = \sqrt{(-2)^2+1^2} = \sqrt{5}$ Ans

& $\vec{BD} = \vec{D} - \vec{B}$

$\Rightarrow \vec{BD} = (2i+j) - (i-j+2k)$

$= 2i+j-i+j-2k$

$\vec{BD} = i+2j-2k$

$\Rightarrow |\vec{BD}| = \sqrt{1^2+2^2+(-2)^2}$

$= \sqrt{1+4+4}$

$= \sqrt{9}$

$= 3$ Ans

Q:7 Find the direction cosines of $\vec{CB}$ and $\vec{AC}$

Sol

$\vec{CB} = \vec{D} - \vec{C}$

$= (2i+j) - (j+k)$

$= 2i+j-j-k$

$= 2i - 1k = 2i+0j-1k$

$a=2 \quad b=0 \quad c=-1$

Then $d = \sqrt{(2)^2+0^2+(-1)^2} = \sqrt{4+0+1} = \sqrt{5}$

Then the direction cosines are

$$\left.\begin{aligned} \cos\alpha &= \frac{a}{d} = \frac{2}{\sqrt{5}} \\ \cos\beta &= \frac{b}{d} = \frac{0}{\sqrt{5}} \\ \cos\gamma &= \frac{c}{d} = \frac{-1}{\sqrt{5}} \end{aligned}\right\} \text{Ans}$$

Now $\vec{AC} = \vec{C} - \vec{A}$

$= (j+k) - (i+j+k)$

$= j+k-i-j-k$

$= -i$

$= -1i+0j+0k$

$\Rightarrow d = \sqrt{(-1)^2+0^2+0^2} = 1$

Now $\cos\alpha = \frac{a}{d}$, $\cos\beta = \frac{b}{d}$, $\cos\gamma = \frac{c}{d}$

$\Rightarrow \cos\alpha = \frac{-1}{1}$ $\quad \cos\beta = \frac{0}{1}$ $\quad \Rightarrow \cos\gamma = \frac{0}{1}$

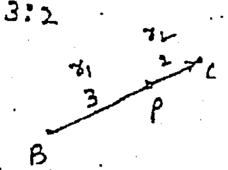
$\Rightarrow \boxed{\cos\alpha = -1}$ Ans $\quad \Rightarrow \boxed{\cos\beta = 0}$ Ans $\quad \Rightarrow \boxed{\cos\gamma = 0}$ Ans

Q:8 Find the position vector of a point which

(i) divides $\vec{BC}$ internally in the ratio 3:2

Sol $r_1 : r_2 = 3:2$

From the ratio theorem



$\vec{OP} = \frac{r_1 c + r_2 b}{r_1+r_2}$

$\Rightarrow \vec{OP} = \frac{3(j+k)+2(i-j+2k)}{3+2}$

$\vec{OP} = \frac{3j+3k+2i-2j+4k}{5}$

$\Rightarrow \vec{OP} = \frac{2i+j+7k}{5}$ Ans

8

(ii) divides $\overrightarrow{AC}$ externally in the ratio 3:2

Sol: By ratio theorem for external point.

$$\overrightarrow{OP} = \frac{r_1 C - r_2 A}{r_1 - r_2} = \frac{3(C) - 2(A)}{3-2}$$

$$= \frac{3(j+k) - 2(i+j+k)}{1}$$

$$= 3j + 3k - 2i - 2j - 2k$$

$$= -2i + 1j + 1k.$$

Q:9 Determine whether any of the following pairs of lines are parallel?

(i) AB & CD

Sol: $\overrightarrow{AB} = B - A$ & $\overrightarrow{CD} = \vec{D} - \vec{C}$

$= (i-j+2k) - (i+j+k)$ $\quad$ $= (2i+j) - (j+k)$

$= i-j+2k-i-j-k$ $\quad$ $= 2i+j-j-k$

$\overrightarrow{AB} = -2j + 1k$ $\quad$ $\overrightarrow{CD} = 2i - k$

Since $AB \neq \lambda \overrightarrow{CD} \Rightarrow AB$ is not parallel to $\overrightarrow{CD}$.

(ii) $\overrightarrow{AC}$ & $\overrightarrow{BD}$

Sol: $\overrightarrow{AC} = \vec{C} - \vec{A}$ & $\overrightarrow{BD} = \vec{D} - \vec{B}$

$= (j+k) - (i+j+k)$ $\quad$ $= (2i+j) - (i-j+2k)$

$= j+k-i-j-k$ $\quad$ $= 2i+j-i+j-2k$

$= -i$ $\quad$ $= i+2j-2k$

Since $\overrightarrow{AC} \neq \lambda \overrightarrow{BD}$

$\Rightarrow \overrightarrow{AC}$ is not parallel to $\overrightarrow{BD}$.

(iii) $\overrightarrow{AD}$ & $\overrightarrow{BC}$

Sol: $\overrightarrow{AD} = \vec{D} - \vec{A}$ & $\overrightarrow{BC} = \vec{C} - \vec{B}$

$= (2i+j) - (i+j+k)$ $\quad$ $= (j+k) - (i-j+2k)$

$= 2i+j-i-j-k$ $\quad$ $= j+k-i+j-2k$

$= i - k$ $\quad$ $= -i+2j-k$

Since $\overrightarrow{AD} \neq \lambda \overrightarrow{BC}$

$\Rightarrow \overrightarrow{AD}$ is not parallel to $\overrightarrow{BC}$.

Self question:

$E = (2,3,4)$ $\quad$ $F = (4,6,5)$ $\quad$ $G = (5,6,9)$ $\quad$ $H = (9,12,11)$

$\Rightarrow \overrightarrow{EF} = (4-2, 6-3, 5-4)$

$= (2,3,1)$

$\overrightarrow{EF} = 2i+3j+1k$

$\overrightarrow{GH} = (9-5, 12-6, 11-9)$

$= (4,6,2)$

$= 4i+6j+2k$

$\overrightarrow{GH} = 2(2i+3j+1k)$

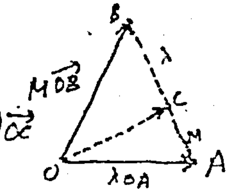
Since $GH = 2(2i+3j+1k)$.

$\overrightarrow{GH} = 2\ \overrightarrow{EF}$

$\Rightarrow \overrightarrow{GH} = \lambda \overrightarrow{EF} \Rightarrow \overrightarrow{GH} \parallel \overrightarrow{EF}$

λ, μ Theorem: If two concurrent forces are $\lambda \overrightarrow{OA}$ and $\mu \overrightarrow{OB}$, their resultant is $(\lambda+\mu)\overrightarrow{OC}$ where C divides AB so that $AC : CB = \mu : \lambda$

$\lambda \overrightarrow{OA} + \mu \overrightarrow{OB} = (\lambda+\mu)\overrightarrow{OC}$



If $\lambda = \mu = 1$ (C is mid point of AB)

$\overrightarrow{OA} + \overrightarrow{OB} = 2\overrightarrow{OC}$

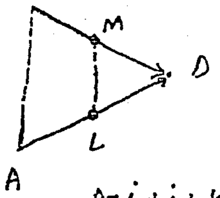
$\Rightarrow \overrightarrow{OC} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$

Q:10 If L and M are position vectors of mid points of $\overrightarrow{AB}$ and $\overrightarrow{BD}$ respectively, show that $\overrightarrow{LM}$ is parallel to $\overrightarrow{AB}$.

Sol

L is mid point of $\overrightarrow{AD}$

M " " " " $\overrightarrow{BD}$



$\overrightarrow{OA} = i+j+k$, $\overrightarrow{OD} = 2i+j$

$A = i+j+k$
$B = i-j+2k$
$C = j+k$
$D = 2i+j$

By (λ, M) Theorem (see page # 06)

$$\overrightarrow{OL} = \frac{\overrightarrow{OA}+\overrightarrow{OD}}{2} = \frac{(i+j+k)+(2i+j)}{2}$$

$$= \frac{3i+2j+k}{2}$$

$$\overrightarrow{OM} = \frac{\overrightarrow{OB}+\overrightarrow{OD}}{2} = \frac{(i-j+2k)+(2i+j)}{2} = \frac{3i+2k}{2}$$

Now $\overrightarrow{LM} = \overrightarrow{OM} - \overrightarrow{OL}$

$$= \frac{3i+2k}{2} - \frac{3i+2j+k}{2}$$

$$= \frac{3i+2k-3i-2j-k}{2}$$

$$\overrightarrow{LM} = \frac{-2j+k}{2} \longrightarrow (i)$$

Now $\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}$

$$= (i-j+2k)-(i+j+k)$$

$$= i-j+2k-i-j-k$$

$$\overrightarrow{AB} = -2j+k$$

Now Eqn (i) $\Rightarrow$ $LM = \frac{-2j+k}{2}$

$$\Rightarrow \overrightarrow{LM} = \frac{1}{2}(-2j+k)$$

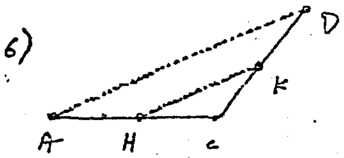
$$\Rightarrow \overrightarrow{LM} = \frac{1}{2}(\overrightarrow{AB})$$

Since $\overrightarrow{LM} = \lambda\,\overrightarrow{AB}$ form

$$\Rightarrow \overrightarrow{LM} \parallel \overrightarrow{AB}$$

Q:11 If H and K are mid points of $\overrightarrow{AC}$ & $\overrightarrow{CD}$, show that $\overrightarrow{HK} = \frac{1}{2}\overrightarrow{AD}$

Sol By λ, M Theorem (see page 6)



$$\overrightarrow{OH} = \frac{\overrightarrow{OA}+\overrightarrow{OC}}{2}$$

$$= \frac{(i+j+k)+(j+k)}{2}$$

$A = i+j+k$
$B = i-j+2k$
$C = j+k$
$D = 2i+j$

$$\overrightarrow{OH} = \frac{i+2j+2k}{2}$$

and $\overrightarrow{OK} = \frac{\overrightarrow{OC}+\overrightarrow{OD}}{2}$

$$= \frac{(j+k)+(2i+j)}{2}$$

$$\Rightarrow \overrightarrow{OK} = \frac{2i+2j+k}{2}$$

Now $\overrightarrow{HK} = \overrightarrow{OK} - \overrightarrow{OH}$

$$= \frac{2i+2j+k}{2} - \frac{i+2j+2k}{2}$$

$$= \frac{2i+2j+k-i-2j-2k}{2}$$

$\Rightarrow \overrightarrow{HK} = \frac{i-k}{2}$

Now $\overrightarrow{AD} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (2i+j)-(i+j+k)$

$= 2i+j-i-j-k$

$\overrightarrow{AD} = i-k$

Now $\overrightarrow{HK} = \frac{i-k}{2}$

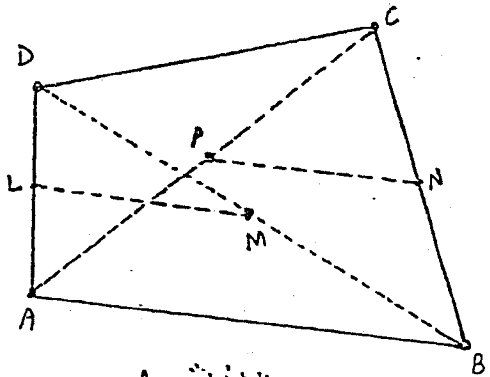
$\Rightarrow \overrightarrow{HK} = \frac{1}{2}(i-k)$

$\Rightarrow \overrightarrow{HK} = \frac{1}{2}\overrightarrow{AD} \Rightarrow \overrightarrow{HK} = \lambda \overrightarrow{AD}$ form

$\Rightarrow \overrightarrow{HK} \parallel \overrightarrow{AD}$

5.8

Q.12 If L, M, N and P are the mid points of $\overrightarrow{AD}$, $\overrightarrow{BD}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ respectively. Show that $\overrightarrow{LM}$ is parallel to $\overrightarrow{NP}$.



$A = i+j+k$

$B = i-j+2k$

$C = j+k$

$D = 2i+j$

Sol

$\overrightarrow{AC} = \vec{C} - \vec{A}$

$= (j+k)-(i+j+k)$

$= j+k-i-j-k$

$= -i$

$\overrightarrow{BD} = \vec{D} - \vec{B}$

$= (2i+j)-(i-j+2k)$

$= (2i+j-i+j-2k)$

$= i+2j-2k$

L is mid point of $\overrightarrow{AD}$, & M is mid point of $\overrightarrow{BD}$

By λ, μ Theorem

$\overrightarrow{OL} = \frac{\overrightarrow{OA}+\overrightarrow{OD}}{2}$

$= \frac{(i+j+k)+(2i+j)}{2}$

$= \frac{3i+2j+k}{2}$

$\overrightarrow{OM} = \frac{\overrightarrow{OB}+\overrightarrow{OD}}{2}$

$= \frac{(i-j+2k)+(2i+j)}{2}$

$= \frac{3i+2k}{2}$

Now N is mid point of BC & P is mid point of $\overrightarrow{AC}$

$\overrightarrow{ON} = \frac{\overrightarrow{OB}+\overrightarrow{OC}}{2}$

$= \frac{(i-j+2k)+(j+k)}{2}$

$\overrightarrow{ON} = \frac{i+3k}{2}$

$\overrightarrow{OP} = \frac{\overrightarrow{OA}+\overrightarrow{OC}}{2}$

$\overrightarrow{OP} = \frac{(i+j+k)+(j+k)}{2}$

$\overrightarrow{OP} = \frac{i+2j+2k}{2}$

Now $\overrightarrow{LM} = \overrightarrow{OM} - \overrightarrow{OL}$

$= \frac{3i+2k}{2} - \frac{3i+2j+k}{2}$

$= \frac{3i+2k-3i-2j-k}{2}$

$\Rightarrow \overrightarrow{LM} = \frac{-2j+k}{2}$

& $\overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON}$

$= \frac{i+2j+2k}{2} - \frac{i+3k}{2}$

$= \frac{i+2j+2k-i-3k}{2}$

$\Rightarrow \overrightarrow{NP} = \frac{2j-k}{2}$

Now $\overrightarrow{LM} = -\left(\frac{2j-k}{2}\right)$

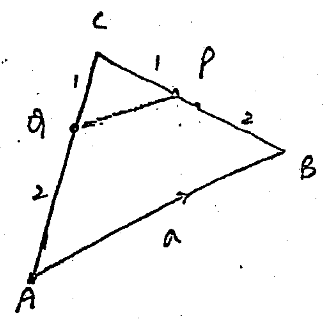
$\overrightarrow{LM} = -\overrightarrow{NP}$

$\Rightarrow \overrightarrow{LM} = \lambda \overrightarrow{NP}$

$\Rightarrow LM \parallel NP$

Q.13 Let P and Q divide the sides $\overrightarrow{BC}$ and $\overrightarrow{AC}$ respectively of ΔABC in the ratio 2:1. If $\overrightarrow{AB} = a$ and $\overrightarrow{AC} = b$, then find $\overrightarrow{QP}$ and hence show that $\overrightarrow{QP}$ is parallel to $\overrightarrow{AB}$ and is one third of its length.

Sol



Suppose $\overrightarrow{OA} = a$

$\overrightarrow{OB} = b$

$\overrightarrow{OC} = c$

Position vector of P

$$\overrightarrow{OP} = \frac{2c+1b}{2+1} = \frac{b+2c}{3} \longrightarrow (1)$$

Now position vector of Q

$$\overrightarrow{OQ} = \frac{1(a)+2(c)}{1+2} = \frac{a+2c}{3} \longrightarrow (2)$$

Now $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$

$$= \frac{b+2c}{3} - \frac{a+2c}{3}$$

$$= \frac{b+2c-a-2c}{3} = \frac{b-a}{3}$$

$$\overrightarrow{QP} = \frac{b-a}{3} \longrightarrow (3)$$

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= b - a$

Eqn (3) $\Rightarrow$ $\overrightarrow{QP} = \frac{\overrightarrow{AB}}{3}$ Hence $\overrightarrow{QP}$ = one third of AB

and $\overrightarrow{QP} \parallel \overrightarrow{AB}$.

Q.14 Find the coordinates of P where

(a) $|\overrightarrow{OP}| = 6$ and $\overrightarrow{OP}$ is in the direction of $2i-3j+6k$.

Sol $\vec{V} = 2i-3j+6k$

$$\Rightarrow |\vec{V}| = \sqrt{2^2+(-3)^2+6^2} = \sqrt{4+9+36} = \sqrt{49} = 7$$

Now $$\hat{V} = \frac{\vec{V}}{|V|} = \frac{2i-3j+6k}{7}$$

Also $(\hat{OP}) = \hat{V}$ ($\because$ same direction)

$$\Rightarrow \hat{OP} = \frac{2i-3j+6k}{7}$$

Then $\overrightarrow{OP} = |\overrightarrow{OP}|\,(\hat{OP})$

$$= 6\left(\frac{2i-3j+6k}{7}\right)$$

$$= \frac{12i-18j+36k}{7}$$

$$= \frac{12}{7}i - \frac{18}{7}j + \frac{36}{7}k$$

Hence $\left(\frac{12}{7}, -\frac{18}{7}, \frac{36}{7}\right)$ are the required coordinates of P.

ENGR.MAJID AMIN
BSc, Mechanical Engineering
from U.E.T Peshawar

(b) $|\overrightarrow{OP}| = 2$ and $\overrightarrow{OP}$ is in the direction of $8i+j-4k$.

Sol $\vec{V} = 8i+j-4k$

$$|V| = \sqrt{8^2+1^2+(-4)^2}$$

$$= \sqrt{64+1+16} = \sqrt{81} = 9$$

Then $\hat{V} = \dfrac{\vec{V}}{|V|} = \dfrac{8i+1j-4k}{9}$

Then $(\hat{OP}) = \hat{V}$

$\Rightarrow (\hat{OP}) = \dfrac{8i+1j-4k}{9}$

Now $\overrightarrow{OP} = |\overrightarrow{OP}|\,(\hat{OP})$

$\Rightarrow \overrightarrow{OP} = 2\left(\dfrac{8i+1j-4k}{9}\right) = \dfrac{16i+2j-8k}{9} = \dfrac{16}{9}i+\dfrac{2}{9}j-\dfrac{8}{9}k$

Hence coordinates of P are $\left(\frac{16}{9}, \frac{2}{9}, -\frac{8}{9}\right)$

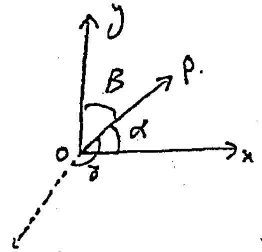
(c) $|\overrightarrow{OP}| = 4$ and $\overrightarrow{OP}$ is inclined at equal acute angles OX, OY and OZ.

Sol Given that $\alpha = \beta = \gamma$

where α is angle of OP with OX

β " " " " " OY

γ " " " " " OZ



As $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ (sum of squares of direction cosines is 1)

$\Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1 \quad \because \beta = \gamma = \alpha$

$\Rightarrow 3\cos^2\alpha = 1$

$\Rightarrow \cos^2\alpha = \frac{1}{3} \Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$

$\Rightarrow \cos\beta = \frac{1}{\sqrt{3}}$ and $\cos\gamma = \frac{1}{\sqrt{3}}$

$\Rightarrow \alpha = \beta = \gamma = \cos^{-1}\frac{1}{\sqrt{3}}$

$= 54.73^\circ$

Then direction ratios will be

$a = \overrightarrow{OP}\cos\alpha = b = c$ $\qquad \cos\alpha = \dfrac{x}{|OP|}$

$a = 4\left(\frac{1}{\sqrt{3}}\right)$

Hence $a = b = c = \frac{4}{\sqrt{3}}$

Hence coordinates of P are $\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$ A

(15) Find the magnitude and inclination to each of the coordinate axes of the vector V, if

(a) $V = 3i+4j+5k$. $\qquad a=3,\ b=4,\ c=5$

Sol $|V| = \sqrt{3^2+4^2+5^2} = \sqrt{50} = 5\sqrt{2}$ $\qquad d = |V| = 5\sqrt{2}$

Now $\cos\alpha = \dfrac{a}{d} = \dfrac{3}{5\sqrt{2}} \Rightarrow \alpha = \cos^{-1}\dfrac{3}{5\sqrt{2}} = 64.9^\circ$

$\cos\beta = \dfrac{b}{d} = \dfrac{4}{5\sqrt{2}} \Rightarrow \beta = \cos^{-1}\dfrac{4}{5\sqrt{2}} = 55.5^\circ$

$\cos\gamma = \dfrac{c}{d} = \dfrac{5}{5\sqrt{2}} \Rightarrow \gamma = \cos^{-1}\dfrac{1}{\sqrt{2}} = 45^\circ$

(b) $\vec{V} = -i + j - k$

$|V| = \sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{3}$

Now $\cos\alpha = \frac{-1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

$\cos\beta = \frac{1}{\sqrt{3}} \Rightarrow \beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

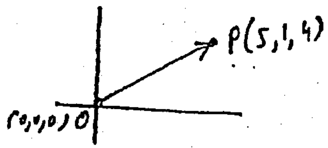
$\cos\gamma = \frac{-1}{\sqrt{3}} \Rightarrow \gamma = \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

(c) v is represented by $\overrightarrow{OP}$ where P is the point $(5,1,4)$.

Sol $\vec{V} = \overrightarrow{OP}$

$= (5,1,4) - (0,0,0)$

$= (5,1,4)$



$\vec{V} = 5i + 1j + 4k$

$\Rightarrow |V| = \sqrt{5^2 + 1^2 + 4^2} = \sqrt{42}$

Now $\cos\alpha = \frac{5}{\sqrt{42}} \Rightarrow \alpha = \cos^{-1}\frac{5}{\sqrt{42}}$

$\cos\beta = \frac{1}{\sqrt{42}} \Rightarrow \beta = \cos^{-1}\frac{1}{\sqrt{42}}$

$\cos\gamma = \frac{4}{\sqrt{42}} \Rightarrow \gamma = \cos^{-1}\frac{4}{\sqrt{42}}$

$\boxed{Q:16}$ If $a = 3i - j - k$

$b = -2i + 4j - 3k$

$c = i + 2j - k$

Then find a unit vector parallel to $3a + 2b + 4c$.

Sol $3a + 2b + 4c$

$= 3(3i - j - k) + 2(-2i + 4j - 3k) + 4(i + 2j - k)$

$= (9i - 3j - 3k) + (-4i + 8j - 6k) + (4i + 8j - 4k)$

$= 9i + 13j - 13k$

Let $\vec{V} = 3a + 2b + 4c$

$\Rightarrow \vec{V} = 9i + 13j - 13k$

Now $|V| = \sqrt{9^2 + 13^2 + (-13)^2}$

$\Rightarrow |V| = \sqrt{81 + 169 + 169}$

$\Rightarrow |V| = \sqrt{419}$

Then the unit vector will be

$\hat{V} = \frac{\vec{V}}{|V|} = \frac{9i + 13j - 13k}{\sqrt{419}}$ Ans

Quote

character is like a tree and reputation like its shadow. The shadow is what we think of it; the tree is the real thing.

Abraham Lincoln (1809 - 1865)

Exercise # 3.4

Q:1 Find the cosine of the angle b/w the vectors
$a = 2i - 8j + 3k$, $b = 4j + 3k$

Sol

$|a| = \sqrt{2^2 + (-8)^2 + 3^2} = \sqrt{4 + 64 + 9} = \sqrt{77}$

$|b| = \sqrt{4^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5$

Now $a \cdot b = (2i - 8j + 3k)\cdot(0i + 4j + 3k)$

$\Rightarrow |a||b|\cos\theta = (2\times 0) + (-8\times 4) + (3\times 3)$

$\Rightarrow \sqrt{77}\; 5 \cos\theta = 0 - 32 + 9$

$\Rightarrow 5\sqrt{77}\cos\theta = -23 \Rightarrow \cos\theta = \dfrac{-23}{5\sqrt{77}}$ Ans

Q:2 The angle b/w two vectors V_1 and V_2 is $\arccos\frac{4}{21}$. If $V_1 = 6i + 3j - 2k$, $V_2 = -2i + \lambda j - 4k$. Find the positive value of λ.

Sol

$|V_1| = \sqrt{6^2 + 3^2 + (-2)^2} = \sqrt{36+9+4} = \sqrt{49} = 7$

$|V_2| = \sqrt{(-2)^2 + \lambda^2 + (-4)^2} = \sqrt{4+\lambda^2+16} = \sqrt{\lambda^2+20}$

Now $V_1 \cdot V_2 = (6i + 3j - 2k)\cdot(-2i + \lambda j - 4k)$

$|V_1||V_2|\cos\theta = -12 + 3\lambda + 8$

$7\sqrt{\lambda^2+20}\;\frac{4}{21} = -4 + 3\lambda$

$\theta = \arccos\frac{4}{21}$
$\theta = \cos^{-1}\frac{4}{21}$
$\cos\theta = 4/21$

$\Rightarrow \frac{4}{3}\sqrt{\lambda^2+20} = -4 + 3\lambda$

$\Rightarrow 4\sqrt{\lambda^2+20} = -12 + 9\lambda$

squaring b.s

$\Rightarrow 16(\lambda^2+20) = (-12 + 9\lambda)^2$

$\Rightarrow 16\lambda^2 + 320 = (-12)^2 + (9\lambda)^2 - 2(12)(9\lambda)$

$\Rightarrow 16\lambda^2 + 320 = 144 + 81\lambda^2 - 216\lambda$

$\Rightarrow 0 = 81\lambda^2 - 16\lambda^2 - 216\lambda + 144 - 320$

$\Rightarrow 65\lambda^2 - 216\lambda - 176 = 0$

By quadratic formula, we get

$\lambda = \dfrac{-(-216) \pm \sqrt{(216)^2 - 4(65)(-176)}}{2(65)}$

$\Rightarrow \lambda = \dfrac{+216 \pm \sqrt{46656 + 45760}}{130}$

$\Rightarrow \lambda = \dfrac{+216 \pm \sqrt{92416}}{130}$

$\Rightarrow \lambda = \dfrac{+216 \pm 304}{130}$

$\lambda = \dfrac{+216 + 304}{130}$ & $\lambda = +\dfrac{216 - 304}{2}$

$\lambda = 4$ & $\lambda = -ve$

Hence the positive value of λ is 4

Q:3 If $a = 3i + 4j - k$, $b = i - j + 3k$, $c = 2i + j - 5k$
Then find

(a) $a \cdot b$

Sol $a \cdot b = (3i + 4j - k) \cdot (i - j + 3k)$
$= 3 - 4 - 3$
$= -4$ Ans

(b) $a \cdot c$

$a \cdot c = (3i + 4j - k) \cdot (2i + j - 5k)$
$\Rightarrow a \cdot c = 6 + 4 + 5 = 15$ Ans

(c) $a \cdot (b + c)$

Sol $a \cdot (b+c) = (3i + 4j - k) \cdot \{(i - j + 3k) + (2i + j - 5k)\}$
$= (3i + 4j - k) \cdot (3i + 0j - 2k)$
$= (3 \times 3) + (4 \times 0) + (-1 \times -2)$
$= 9 + 0 + 2$
$= 11$ Ans

(d) $(2a + 3b) \cdot c$

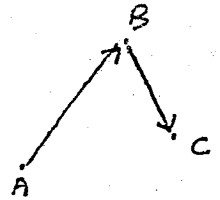
Sol $(2a + 3b) \cdot c = \{2(3i + 4j - k) + 3(i - j + 3k)\} \cdot (2i + j - 5k)$
$= \{(6i + 8j - 2k) + (3i - 3j + 9k)\} \cdot (2i + j - 5k)$
$= (9i + 5j + 7k) \cdot (2i + j - 5k)$
$= 18 + 5 - 35$
$= -12$ Ans

(e) $(a - b) \cdot c$

Sol $(a - b) \cdot c = \{(3i + 4j - k) - (i - j + 3k)\} \cdot (2i + j - 5k)$
$= (3i + 4j - k - i + j - 3k) \cdot (2i + j - 5k)$
$= (2i + 5j - 4k) \cdot (2i + j - 5k)$
$= 4 + 5 + 20$
$= 29$ Ans

Q:4 In ΔABC, $\overrightarrow{AB} = i + 2j + 3k$, $\overrightarrow{BC} = -4i + 4j$

(a) Find the cosine of $\angle ABC$.



Sol $\overrightarrow{AB} = (1, 2, 3) \Rightarrow |AB| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$
$\overrightarrow{BC} = (-4, 4, 0) \Rightarrow |BC| = \sqrt{(-4)^2 + 4^2} = \sqrt{32}$

Now $\overrightarrow{AB} \cdot \overrightarrow{BC} = (i + 2j + 3k) \cdot (-4i + 4j + 0k)$

$\Rightarrow |AB|\,|BC| \cos\angle ABC = -4 + 8 + 0$

$\Rightarrow \sqrt{14}\sqrt{32} \cos\angle ABC = 4$

$\Rightarrow \cos\angle ABC = \frac{4}{\sqrt{14}\sqrt{32}} = \frac{4}{\sqrt{14 \times 32}} = \frac{4}{\sqrt{448}} = \frac{4}{\sqrt{16 \times 28}}$

$\cos\angle ABC = \frac{4}{4\sqrt{28}} \Rightarrow \cos\angle ABC = \frac{1}{\sqrt{28}}$ Ans

(b) Find the vector $\overrightarrow{AC}$ and use it to calculate angle $\angle BAC$.

Sol $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
$= (1, 2, 3) + (-4, 4, 0)$
$= (-3, 6, 3)$

$\Rightarrow \overrightarrow{AC} = -3i + 6j + 3k$

$\Rightarrow |\overrightarrow{AC}| = \sqrt{(-3)^2 + 6^2 + 3^2}$

$= \sqrt{9+36+9} = \sqrt{54} = \sqrt{9\times 6} = 3\sqrt{6}.$

Now $\cos\angle BAC = \dfrac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{|AC|\,|AB|}$

$= \dfrac{(-3i+6j+3k)\cdot(1i+2j+3k)}{3\sqrt{6}\ \sqrt{14}}$

$= \dfrac{-3+12+9}{3\sqrt{14\times 6}}$

$= \dfrac{18}{3\sqrt{14}\sqrt{6}} = \dfrac{6}{\sqrt{14}\sqrt{6}} = \dfrac{\sqrt{6}\sqrt{6}}{\sqrt{14}\sqrt{6}} = \sqrt{\dfrac{6}{14}} = \sqrt{\dfrac{3}{7}}$

Hence $\cos\angle BAC = \sqrt{3/7}$.

$\Rightarrow \angle BAC = \cos^{-1}\sqrt{3/7}$

Q:5 A, B, C are points with position vectors a, b and c respectively, relative to the origin O. AB is perpendicular to $\overrightarrow{OC}$ and $\overrightarrow{BC}$ is perpendicular to $\overrightarrow{OA}$. Show that $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{OB}$.

Sol Given that $\overrightarrow{OA} = a,\ \overrightarrow{OB} = b,\ \overrightarrow{OC} = c$

and $\overrightarrow{AB} \perp \overrightarrow{OC}$ & $\overrightarrow{BC} \perp \overrightarrow{OA}$.

To show $\overrightarrow{AC} \perp \overrightarrow{OB}$.

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \vec{c} - \vec{a}$

and $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{c} - \vec{b}$

Since $\overrightarrow{AB} \perp \overrightarrow{OC} \Rightarrow \overrightarrow{AB}\cdot\overrightarrow{OC} = 0$

$\Rightarrow (b-a)\cdot c = 0$

$\Rightarrow \vec{b}\cdot\vec{c} - \vec{a}\cdot\vec{c} = 0 \longrightarrow (i)$

Also $\overrightarrow{BC} \perp \overrightarrow{OA} \Rightarrow \overrightarrow{BC}\cdot\overrightarrow{OA} = 0$

$\Rightarrow (\vec{c} - \vec{b})\cdot\vec{a} = 0$

$\Rightarrow \vec{c}\cdot\vec{a} - \vec{b}\cdot\vec{a} = 0 \longrightarrow (ii)$

Adding eqn (i) and (ii), we get

Eqn (i) $\Rightarrow \vec{b}\cdot\vec{c} - \vec{a}\cdot\vec{c} = 0$

Eqn (ii) $\Rightarrow -\vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{c} = 0$

$\vec{b}\cdot\vec{c} - \vec{b}\cdot\vec{a} = 0$

$\Rightarrow \vec{b}\cdot(\vec{c} - \vec{a}) = 0$

$\Rightarrow (\overrightarrow{OB})\cdot(\overrightarrow{AC}) = 0$

$\Rightarrow \overrightarrow{OB} \perp \overrightarrow{AC}$

Q:6 Given two vectors $\vec{a}, \vec{b}$ $(a \neq 0, \vec{b} \neq 0)$ Show that

(a) If $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular then $|\vec{a}| = |\vec{b}|$

(b) $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ then $\vec{a} \perp \vec{b}$.

Sol (a) $(\vec{a}+\vec{b}) \perp (\vec{a}-\vec{b})$

then $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 0$

$\Rightarrow a^2 - b^2 = 0 \Rightarrow a^2 = b^2 \Rightarrow \boxed{|a| = |b|}$

(b) $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

squaring b.s, we get

$\Rightarrow |a+b|^2 = |a-b|^2$

$\Rightarrow a^2+b^2+2ab = a^2+b^2-2ab$

$\Rightarrow 2ab = -2ab \Rightarrow 4ab = 0 \Rightarrow ab=0$

Since $ab=0 \Rightarrow a.b=0 \Rightarrow \vec{a} \perp \vec{b}$. Hence proved.

Q:7 Three vectors a, b and c are such that $a \neq b \neq c$ and none is zero.

(a) If $\vec{a}.(\vec{b}+\vec{c}) = \vec{b}.(\vec{a}-\vec{c})$. Prove that $\vec{c}.(\vec{a}+\vec{b})=0$

Sol

$\vec{a}.(\vec{b}+\vec{c}) = \vec{b}.(\vec{a}-\vec{c})$

$\Rightarrow \vec{a}.\vec{b} + \vec{a}.\vec{c} = \vec{b}.\vec{a} - \vec{b}.\vec{c}$

$\Rightarrow \vec{a}.\vec{b} + \vec{a}.\vec{c} = \vec{a}.\vec{b} - \vec{b}.\vec{c} \quad \because \vec{a}.\vec{b} = \vec{b}.\vec{a}$

$\Rightarrow \vec{a}.\vec{c} = -\vec{b}.\vec{c}$

$\Rightarrow \vec{a}.\vec{c} + \vec{b}.\vec{c} = 0$

$\Rightarrow \vec{c}.(\vec{a}+\vec{b}) = 0$ Hence proved.

(b) If $(\vec{a}.\vec{b})\vec{c} = (\vec{b}.\vec{c})\vec{a}$ ① Show that a and c are parallel.

Sol As we know that the dot product of two vectors is a constant (scalar)

let $\vec{a}.\vec{b} = k$, $\vec{b}.\vec{c} = t$

Then eqn ① $\Rightarrow k\vec{c} = t\vec{a} \Rightarrow \vec{c} = \frac{t}{k}\vec{a}$

Since $\vec{c}$ & $\vec{a}$ are multiple of each other $\Rightarrow \vec{a} \parallel \vec{c}$.

Q:8 Find the angle between the following pairs of vectors.

(a) $r_1 = i+2j-k$

$r_2 = i+j-2k$

Sol $|r_1| = \sqrt{1^2+2^2+(-1)^2} = \sqrt{1+4+1} = \sqrt{6}$

$|r_2| = \sqrt{1^2+(1)^2+(-2)^2}$

$|r_2| = \sqrt{1+1+4}$

$|r_2| = \sqrt{6}$

Now $\vec{r_1}.\vec{r_2} = (i+2j-k).(i+j-2k)$

$\Rightarrow |r_1||r_2|\cos\theta = 1+2+2$

$\Rightarrow \sqrt{6}\sqrt{6}\cos\theta = 5$

$\Rightarrow 6\cos\theta = 5 \Rightarrow \cos\theta = 5/6$

$\Rightarrow \theta = \cos^{-1}(\frac{5}{6})$ Ans

(b) $r_1 = \lambda(i+2j+2k)$, $r_2 = \mu(3i+2j+6k)$

Sol $r_1 = \lambda i + 2\lambda j + 2\lambda k$

$\Rightarrow |r_1| = \sqrt{\lambda^2+(2\lambda)^2+(2\lambda)^2}$

$\Rightarrow |r_1| = \sqrt{9\lambda^2} = 3\lambda$

$r_2 = 3\mu i + 2\mu j + 6\mu k$

$|r_2| = \sqrt{(3\mu)^2+(2\mu)^2+(6\mu)^2}$

$|r_2| = \sqrt{9\mu^2+4\mu^2+36\mu^2}$

$\Rightarrow |r_2| = \sqrt{49\mu^2}$

$\Rightarrow |r_2| = 7\mu$

ENGR.MAJID AMIN
BSc, Mechanical Engineering
from U.E.T Peshawar

Now $r_1 \cdot r_2 = (\lambda i + 2\lambda j + 2\lambda k)\cdot(3\mu i + 2\mu j + 6\mu k)$

$\Rightarrow |r_1|\,|r_2| \cos\theta = 3\lambda\mu + 4\lambda\mu + 12\lambda\mu$

$\Rightarrow (3\lambda)(7\mu)\cos\theta = 19\lambda\mu$

$\Rightarrow 21\lambda\mu\cos\theta = 19\lambda\mu \Rightarrow \cos\theta = \frac{19}{21} \Rightarrow \boxed{\theta = \cos^{-1}\frac{19}{21}}$ Ans

Q:9 Show that $i + 7j + 3k$ is perpendicular to both $i - j + 2k$ and $2i + j - 3k$.

Sol $\vec{v_1} = i + 7j + 3k$

$\vec{v_2} = i - j + 2k$

$\vec{v_3} = 2i + j - 3k$

Finding $\vec{v_1}\cdot\vec{v_2} = (i + 7j + 3k)\cdot(i - j + 2k)$

$\Rightarrow \vec{v_1}\cdot\vec{v_2} = 1 - 7 + 6$

$\Rightarrow \vec{v_1}\cdot\vec{v_2} = 0 \Rightarrow \vec{v_1} \perp \vec{v_2}$.

Now $\vec{v_1}\cdot\vec{v_3} = (i + 7j + 3k)\cdot(2i + j - 3k)$

$\Rightarrow \vec{v_1}\cdot\vec{v_3} = 2 + 7 - 9$

$\Rightarrow \vec{v_1}\cdot\vec{v_3} = 0 \Rightarrow \vec{v_1} \perp \vec{v_3}$.

Q:10 Show that $13i + 23j + 7k$ is perpendicular to both $2i + j - 7k$ and $3i - 2j + k$.

Sol $\vec{v_1} = 13i + 23j + 7k$

$\vec{v_2} = 2i + j - 7k$

$\vec{v_3} = 3i - 2j + k$

First $\vec{v_1}\cdot\vec{v_2} = (13i + 23j + 7k)\cdot(2i + j - 7k)$

$= 26 + 23 - 49$

$= 49 - 49 = 0$

$\Rightarrow \vec{v_1} \perp \vec{v_2}$

Now $\vec{v_1}\cdot\vec{v_3} = (13i + 23j + 7k)\cdot(3i - 2j + k)$

$= 39 - 46 + 7$

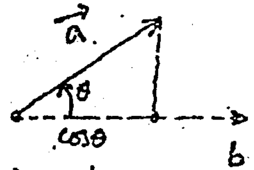
$= 0$

$\Rightarrow \vec{v_1} \perp \vec{v_3}$.

Q:11 Find the projection of $3i + j - 2k$ on $-i - j + 5k$.

Sol Let $\vec{a} = 3i + j - 2k$

$\vec{b} = -i - j + 5k$



$\Rightarrow \frac{a\cdot b}{|b|}$

Sol $|b| = \sqrt{(-1)^2 + (-1)^2 + 5^2}$

$|b| = \sqrt{1 + 1 + 25} = \sqrt{27}$

Now $a\cdot b = (3i + j - 2k)\cdot(-i - j + 5k)$

$\Rightarrow a\cdot b = -3 - 1 - 10$

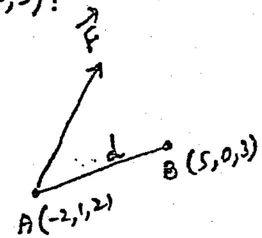
$\Rightarrow a\cdot b = -14$

The projection of a on b $= a\cos\theta = \frac{a\cdot b}{|b|} = \frac{-14}{\sqrt{27}}$

$= \frac{-14}{3\sqrt{3}}$ Ans

Q:12 Find the work done by the force $\vec{F} = 2i + 3j + k$ in displacement of an object from a point $A(-2,1,2)$ to the point $B(5,0,3)$?

Sol Given that $\vec{F} = 2i + 3j + 1k$



Now d = Head − Tail

$d = (5,0,3) - (-2,1,2)$

$d = (7,-1,1)$

$d = 7i - 1j + 1k$

Now work done $= W = F \cdot d$

$= (2i + 3j + 1k) \cdot (7i - 1j + 1k)$

$= 14 - 3 + 1$

$\boxed{W = 12 \text{ units}}$

Engr. Majid Amin
MS. Mechanical Engineering
UET. Peshawar.

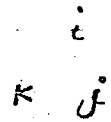
Exercise # 3.5

Q:1 (i) $j \times (2j + 3k)$

$= 2\, j \times j + 3\, j \times k$

$= 2(0) + 3\, i$

$= 3i$



(ii) $(2i - 3j) \times k$

$= 2(i \times k) - 3(j \times k)$

$= 2(-j) - 3i$

$= -2j - 3i$

$= -3i - 2j$

(iii) $\vec{a} = 2i - 3j + 5k$
$\vec{b} = 6i + 2j - 3k$ Find $\vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & -3 & 5 \\ 6 & 2 & -3 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = i\begin{vmatrix} -3 & 5 \\ 2 & -3 \end{vmatrix} - j\begin{vmatrix} 2 & 5 \\ 6 & -3 \end{vmatrix} + k\begin{vmatrix} 2 & -3 \\ 6 & 2 \end{vmatrix}$$

$= i(9-10) - j(-6-30) + k(4+18)$

$= i(-1) - j(-36) + k(22)$

$= -i + 36j + 22k$

Q:2 $a = -2i + 6j + 3K$
$b = 3i + 3j + 6K$
$c = 2i + 7j + 4K$

Find $(a-b)\times(c-a)$ and $(a+b)\times(c-a)$.

Sol $a-b = (-2i+6j+3K) - (3i+3j+6K)$
$= -5i + 3j - 3K$

& $c-a = (2i+7j+4K) - (-2i+6j+3K)$
$= 4i + 1j + 1K$

Now $(a-b)\times(c-a) = \begin{vmatrix} i & j & K \\ -5 & 3 & -3 \\ 4 & 1 & 1 \end{vmatrix}$

$= i\begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix} - j\begin{vmatrix} -5 & -3 \\ 4 & 1 \end{vmatrix} + K\begin{vmatrix} -5 & 3 \\ 4 & 1 \end{vmatrix}$

$= i(3+3) - j(-5+12) + K(-5-12)$

$= i(6) - j(7) + K(-17)$

$\Rightarrow (a-b)\times(c-a) = 6i - 7j - 17K$ Ans

Now $a+b = (-2i+6j+3K) + (3i+3j+6K)$
$a+b = i + 9j + 9K$

and $c-a = 4i + 1j + 1K$

Now $(a+b)\times(c-a) = \begin{vmatrix} i & j & K \\ 1 & 9 & 9 \\ 4 & 1 & 1 \end{vmatrix}$

$= i\begin{vmatrix} 9 & 9 \\ 1 & 1 \end{vmatrix} - j\begin{vmatrix} 1 & 9 \\ 4 & 1 \end{vmatrix} + K\begin{vmatrix} 1 & 9 \\ 4 & 1 \end{vmatrix}$

$= i(9-9) - j(1-36) + K(1-36) = 0i + 35j - 35K$.

Q:3: Find a unit vector perpendicular to both $\vec{a} = i+j+2K$ and $b = -2i+j-3K$

Sol $\vec{a}\times\vec{b} = \begin{vmatrix} i & j & K \\ 1 & 1 & 2 \\ -2 & 1 & -3 \end{vmatrix}$

$\Rightarrow \vec{a}\times\vec{b} = i\begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} - j\begin{vmatrix} 1 & 2 \\ -2 & -3 \end{vmatrix} + K\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}$

$= i(-3-2) - j(-3+4) + K(1+2)$

$\vec{a}\times\vec{b} = -5i - 1j + 3K$

Let $\vec{a}\times\vec{b} = V$ (and V is $\perp$ar to $\vec{a}$ and $\vec{b}$)

i.e $\vec{V} = -5i - 1j + 3K$

then $|V| = \sqrt{(-5)^2 + (-1)^2 + 3^2} = \sqrt{25+1+9} = \sqrt{35}$

Now a unit vector is

$\hat{V} = \frac{\vec{V}}{|V|} = \frac{-5i-1j+3K}{\sqrt{35}}$ Ans

---

Q:4 Find a vector of magnitude 10 and perpendicular to $\vec{a} = 2i-3j+4K$ and $\vec{b} = 4i-2j-4K$.

Sol $\vec{a}\times\vec{b} = \begin{vmatrix} i & j & K \\ 2 & -3 & 4 \\ 4 & -2 & -4 \end{vmatrix}$

$\Rightarrow \vec{a}\times\vec{b} = i\begin{vmatrix} -3 & 4 \\ -2 & -4 \end{vmatrix} - j\begin{vmatrix} 2 & 4 \\ 4 & -4 \end{vmatrix} + K\begin{vmatrix} 2 & -3 \\ 4 & -2 \end{vmatrix}$

$\Rightarrow \vec{a}\times\vec{b} = i(12+8) - j(-8-16) + K(-4+12)$

$\Rightarrow \vec{a}\times\vec{b} = 20i + 24j + 8K$

96

Let $\vec{v} = \vec{a} \times \vec{b}$

$\vec{v} = 20i + 24j + 8K$

Now $\hat{v} = \dfrac{\vec{v}}{|v|}$

$= \dfrac{20i + 24j + 8k}{\sqrt{(20)^2 + (24)^2 + (8)^2}}$

$= \dfrac{20i + 24j + 8k}{\sqrt{1040}} = \dfrac{20i + 24j + 8k}{4\sqrt{65}}$ $\quad \because 1040 = 16 \times 65$

Now we want to find a vector of magnitude 10 and $\perp$ar to a and b.

Let $\vec{v_1}$ is that vector.

$|v_1| = 10$ given

and $\hat{v_1} = \hat{v}$ because v_1 and v are both $\perp$ar to a and b and hence in the same direction.

Then $\vec{v_1} = |v_1| \, \hat{v_1}$

$\vec{v_1} = 10 \left(\dfrac{20i + 24j + 8k}{4\sqrt{65}} \right)$

$\vec{v_1} = 10 \times \dfrac{4(5i + 6j + 2K)}{4\sqrt{65}} = 10 \left(\dfrac{5i + 6j + 2k}{\sqrt{65}} \right)$

$\vec{v_1} = \dfrac{50i + 60j + 20K}{\sqrt{65}}$ Ans

Q:5 For the vectors

$\vec{a} = 2i - 3j - K$

$\vec{b} = i + 4j - 2K$

Prove that

(a) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Sol L.H.S

$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix}$

$\Rightarrow \vec{a} \times \vec{b} = i \begin{vmatrix} -3 & -1 \\ 4 & -2 \end{vmatrix} - j \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + k \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix}$

$= i(6+4) - j(-4+1) + k(8+3)$

$= 10i + 3j + 11K$

R.H.S

$\vec{b} \times \vec{a} = \begin{vmatrix} i & j & k \\ 1 & 4 & -2 \\ 2 & -3 & -1 \end{vmatrix}$

$= i \begin{vmatrix} 4 & -2 \\ -3 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + K \begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix}$

$= i(-4-6) - j(-1+4) + K(-3-8)$

$= -10i - 3j - 11K$

$= -(10i + 3j + 11K)$

$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$

(b) $(a+b)\times(a-b) = -2(a\times b)$

Sol As before $a\times b = 10i + 3j + 11k$

$a+b = (2i-3j-k) + (i+4j-2k)$

$\Rightarrow a+b = 3i + j - 3k$

and $a-b = (2i-3j-k) - (i+4j-2k)$

$= 2i - 3j - k - i - 4j + 2k$

$\Rightarrow a-b = i - 7j + k$

Now $(a+b)\times(a-b) = \begin{vmatrix} i & j & k \\ 3 & 1 & -3 \\ 1 & -7 & 1 \end{vmatrix}$

L.H.S

$= i\begin{vmatrix} 1 & -3 \\ -7 & 1 \end{vmatrix} - j\begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix} + k\begin{vmatrix} 3 & 1 \\ 1 & -7 \end{vmatrix}$

$= i(1-21) - j(3+3) + k(-21-1)$

$= i(-20) - j(6) + k(-22)$

$= -20i - 6j - 22k$

$= -2(10i + 3j + 11k)$

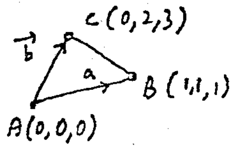
$= -2(a\times b) = R.H.S$

Q:6 Find the area of the triangle ABC whose vertices are $A(0,0,0)$, $B=(1,1,1)$ and $C=(0,2,3)$

Sol $\vec{a} = (1,1,1) - (0,0,0)$ Head - Tail

$= (1,1,1)$

$= 1i + 1j + 1k$



and $\vec{b} = (0,2,3) - (0,0,0)$

$\Rightarrow \vec{b} = (0,2,3) \Rightarrow \vec{b} = 0i + 2j + 3k$

Now $\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix}$

$= i\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$

$= i(3-2) - j(3-0) + k(2-0)$

$\vec{a}\times\vec{b} = 1i - 3j + 2k$

Then $|\vec{a}\times\vec{b}| = \sqrt{(1)^2 + (-3)^2 + 2^2}$

$= \sqrt{1+9+4} = \sqrt{14}$

Then Area of the triangle will be

$A = \frac{1}{2}|\vec{a}\times\vec{b}|$

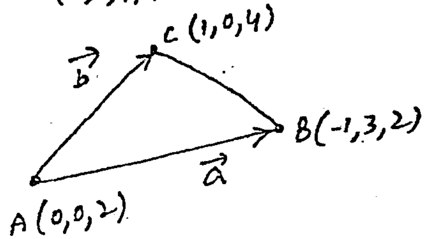
$A = \frac{1}{2}\sqrt{14} \Rightarrow \boxed{A = \frac{\sqrt{14}}{2}}$ Ans units^2

Q:7 Find the area of the triangle whose vertices are $(0,0,2)$, $(-1,3,2)$ and $(1,0,4)$.

Sol $A = (0,0,2)$

$B = (-1,3,2)$

$C = (1,0,4)$



Then $\vec{a} = (-1,3,2) - (0,0,2)$

$= (-1,3,0)$

$\vec{a} = -1i + 3j + 0k$

and $\vec{b} = (1,0,4) - (0,0,2)$

$\vec{b} = (1,0,2)$

$\vec{b} = 1i + 0j + 2k$

Now $\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix}$

$\Rightarrow \vec{a}\times\vec{b} = i\begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} - j\begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix}$

$= i(6-0) - j(-2-0) + k(0-3)$

$= 6i + 2j - 3k$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{6^2 + 2^2 + (-3)^2}$

$= \sqrt{36+4+9} = \sqrt{49} = 7$

So the area of the triangle will be $\frac{1}{2}|a \times b|$

$= \frac{1}{2}(7) = \frac{7}{2}\ \text{unit}^2$

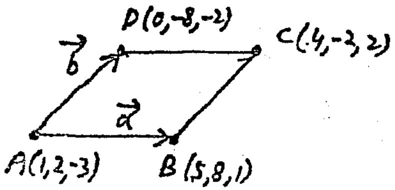
Q:8 Find the area of the parallelogram whose vertices are $A(1,2,-3)$, $B(5,8,1)$, $C(4,-2,2)$ & $D(0,-8,-2)$?

Sol

$\vec{a} = (5,8,1) - (1,2,-3)$

$= (4,6,4)$

$\vec{a} = 4i + 6j + 4k$



& $\vec{b} = (0,-8,-2) - (1,2,-3)$

$= (-1,-10,1)$

$= -1i - 10j + 1k$

Now $\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 4 & 6 & 4 \\ -1 & -10 & 1 \end{vmatrix}$

$= i\begin{vmatrix} 6 & 4 \\ -10 & 1 \end{vmatrix} - j\begin{vmatrix} 4 & 4 \\ -1 & 1 \end{vmatrix} + k\begin{vmatrix} 4 & 6 \\ -1 & -10 \end{vmatrix}$

$= i(6+40) - j(4+4) + k(-40+6)$

$\vec{a}\times\vec{b} = 46i - 8j - 34k$

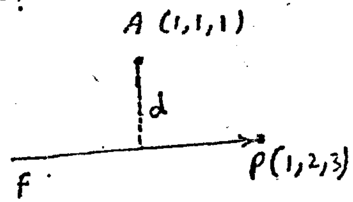
Then area of the parallelogram will be

$A = |a\times b| = \sqrt{(46)^2 + (-8)^2 + (-34)^2}$

$= \sqrt{2116 + 64 + 1156}$

$= \sqrt{3336}\ \text{unit}^2$

Q:9 A force $F = i + 2j - 3k$ is applied at $(1,2,3)$. Find its moment about $(1,1,1)$. What is the magnitude of this moment?



Sol Let $P = (1,2,3)$

$A = (1,1,1)$

Then $\vec{PA} = r = (1,1,1) - (1,2,3)$

$r = (0,-1,-2)$

Then Moment $= M = r \times F$

$= \begin{vmatrix} i & j & k \\ 0 & -1 & -2 \\ 1 & 2 & -3 \end{vmatrix}$

$M = i\begin{vmatrix} -1 & -2 \\ 2 & -3 \end{vmatrix} - j\begin{vmatrix} 0 & -2 \\ 1 & -3 \end{vmatrix} + k\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix}$

$\Rightarrow M = (3+4)i - j(0+2) + k(0+1)$

$\boxed{M = 7i - 2j + 1k}$ Ans

Magnitude

$|M| = \sqrt{(7)^2 + (-2)^2 + 1^2}$

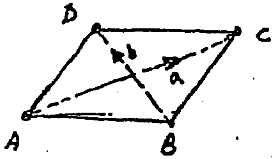
$= \sqrt{49 + 4 + 1}$

$= \sqrt{54}$

Q:10: Find the areas of a parallelogram whose diagnols are

$a = 4i + j - 2k$

$b = -2i + 3j + 4k$



Sol: let $\overrightarrow{AC} = a = (4, 1, -2)$

$\overrightarrow{BD} = b = (-2, 3, 4)$

From the figure

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\vec{a} = \overrightarrow{AB} + \overrightarrow{BC} \longrightarrow (1)$

and $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$

$\vec{b} = \overrightarrow{BA} + \overrightarrow{AD} \longrightarrow (2)$

Adding eqn (1) & (2), we get

$\Rightarrow \vec{a} + \vec{b} = (\overrightarrow{AB} + \overrightarrow{BC}) + (\overrightarrow{BA} + \overrightarrow{AD})$

$\Rightarrow (4,1,-2) + (-2,3,4) = \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AB} + \overrightarrow{AD}$

$\Rightarrow (2,4,2) = BC + AD$ but $AB = AD$

$\Rightarrow (2,4,2) = 2BC$

$\Rightarrow (2,4,2) = 2\overline{BC}$

$\Rightarrow 2i + 4j + 2k = 2BC$

$\div$ by 2

$\Rightarrow \overrightarrow{BC} = i + 2j + k$

Hence $\overrightarrow{BC} = (1,2,1)$

Now Eqn (1) − Eqn (2)

$\Rightarrow a - b = (\overrightarrow{AB} + \overrightarrow{BC}) - (\overrightarrow{BA} + \overrightarrow{AD})$

$a - b = \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{BA} - \overrightarrow{AD}$

$a - b = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AB} - \overrightarrow{AD} \quad \because \overrightarrow{BA} = -\overrightarrow{AB}$

$a - b = 2\overrightarrow{AB} + \cancel{\overrightarrow{BC}} - \cancel{\overrightarrow{AD}} \quad \because \overrightarrow{BC} = \overrightarrow{AD}$

$\Rightarrow (4,1,-2) - (-2,3,4) = 2\overrightarrow{AB}$

$\Rightarrow (6,-2,-6) = 2\overrightarrow{AB}$

$\Rightarrow 6i - 2j - 6k = 2\overrightarrow{AB}$

divide by 2

$\overrightarrow{AB} = 3i - j - 3k$

Now $\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 3 & -1 & -3 \end{vmatrix}$

$= i\begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix} + k\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$

$= i(-6+1) - j(-3-3) + k(-1-6)$

$= -5i + 6j - 7k$

Then area of the parallelogram will be

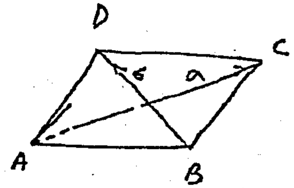
$$\text{Area} = |AB \times BC|$$
$$= \sqrt{(-5)^2 + 6^2 + (-7)^2}$$
$$= \sqrt{25 + 36 + 49}$$
$$= \sqrt{110} \text{ unit}^2$$

(ii) $\vec{a} = 3i + 2j - 2k = (3,2,-2) \longrightarrow ①$

$b = i - 3j + 4k = (1,-3,4) \longrightarrow ②$

Equ ① + Equ ②

$a + b = (3,2,-2) + (1,-3,4)$



$\Rightarrow a + b = (4,-1,2)$

$\Rightarrow (\vec{AB} + \vec{BC}) + (\vec{BA} + \vec{AD}) = (4,-1,2)$

$\Rightarrow \vec{AB} + \vec{BC} - \vec{AB} + \vec{AD} = (4,-1,2)$

$\Rightarrow \vec{BC} + \vec{AD} = (4,-1,2)$ but $\vec{BC} = \vec{AD}$

$\Rightarrow 2\vec{BC} = (4,-1,2)$

$\Rightarrow \vec{BC} = \frac{1}{2}(4,-1,2)$

$\Rightarrow \vec{BC} = (2, \frac{-1}{2}, 1)$

Now Equ ① - Equ ②

$a - b = (3,2,-2) - (1,-3,4)$

$(AB + BC) - (BA + AD) = (2,5,-6)$

$\Rightarrow AB + \cancel{BC} - BA - \cancel{AD} = (2,5,-6)$

$\Rightarrow AB - BA = (2,5,-6)$

$\Rightarrow AB - (-AB) = (2,5,-6)$

$\Rightarrow 2AB = (2,5,-6)$

$\Rightarrow AB = \frac{1}{2}(2,5,-6)$

$\Rightarrow \vec{AB} = (1, \frac{5}{2}, -3)$

Now $\vec{AB} \times \vec{BC} = \begin{vmatrix} i & j & k \\ 1 & 5/2 & -3 \\ 2 & -1/2 & +1 \end{vmatrix}$

$= i\begin{vmatrix} \frac{5}{2} & -3 \\ \frac{-1}{2} & +1 \end{vmatrix} - j\begin{vmatrix} 1 & -3 \\ 2 & +1 \end{vmatrix} + k\begin{vmatrix} 1 & 5/2 \\ 2 & -1/2 \end{vmatrix}$

$= i(\frac{5}{2} - \frac{3}{2}) - j(1+6) + k(-\frac{1}{2} - 5)$

$\vec{AB} \times \vec{BC} = i - 7j - \frac{11}{2}k$

Now Area of the parallelogram will be

$$\text{Area} = |\vec{AB} \times \vec{BC}| = \sqrt{(1)^2 + (-7)^2 + (\tfrac{-11}{2})^2}$$
$$= \sqrt{1 + 49 + \tfrac{121}{4}}$$
$$= \sqrt{\frac{4 + 196 + 121}{4}} = \sqrt{\frac{321}{4}}$$

Hence $\text{Area} = \frac{\sqrt{321}}{2} \text{ unit}^2$

Exercise # 3.6

Q.1 Prove theorem 3 of section 3.27.

(a) For any vectors a, b, c

$$a\cdot(b\times c) = b\cdot(c\times a) = c\cdot(a\times b)$$

(b) The positions of dot and cross in a scalar triple product can be interchanged.

(c) $i\cdot j\times k = j\cdot k\times i = k\cdot i\times j = 1$ &

(d) $i\cdot k\times j = j\cdot i\times k = k\cdot j\times i = -1$

Proofs:

Let
$$a = x_1 i + y_1 j + z_1 k$$
$$b = x_2 i + y_2 j + z_2 k$$
$$c = x_3 i + y_3 j + z_3 k$$

$$a\cdot(b\times c) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \longrightarrow (1)$$

$$= (-)\begin{vmatrix} x_2 & y_2 & z_2 \\ x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad R_1 \leftrightarrow R_2$$

$$= (-)(-)\begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \end{vmatrix}$$

$$= +(b\cdot c\times a) \longrightarrow (2)$$

$$= \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \end{vmatrix}$$

$$= (-)\begin{vmatrix} x_3 & y_3 & z_3 \\ x_2 & y_2 & z_2 \\ x_1 & y_1 & z_1 \end{vmatrix}$$

$$= (-)(-)\begin{vmatrix} x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$= +\ c\cdot(a\times b) \longrightarrow (3)$$

From (1), (2) and (3), it is proved

$$a\cdot(b\times c) = b\cdot(c\times a) = c\cdot(a\times b)$$

(b)
$$a = x_1 i + y_1 j + z_1 k$$
$$b = x_2 i + y_2 j + z_2 k$$
$$c = x_3 i + y_3 i + z_3 k$$

To prove $a\cdot(b\times c) = (a\times b)\cdot c$

R.H.S
$$a\times b = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$\Rightarrow a\times b = i\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - j\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + k\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

$$= i(y_1 z_2 - y_2 z_1) - j(x_1 z_2 - x_2 z_1) + k(x_1 y_2 - x_2 y_1)$$

$$\therefore\ (a\times b)\cdot c = (y_1z_2 - z_1y_2)x_3 - (x_1z_2 - z_1x_2)y_3 + (x_1y_2 - y_1x_2)z_3$$

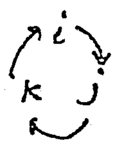
$$= \begin{vmatrix} x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$= -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= a\cdot(b\times c) = L.H.S$$

(c) $i\cdot j\times k = j\cdot k\times i = k\cdot i\times j = 1$



Sol

$i\cdot j\times k = i\cdot i = 1$

$j\cdot k\times i = j\cdot j = 1$

$k\cdot i\times j = k\cdot k = 1$

Hence proved

(d) $i\cdot k\times j = j\cdot i\times k = k\cdot j\times i = -1$

Sol

$i\cdot k\times j = i\cdot -i = -1$

$j\cdot i\times k = j\cdot -j = -1$

$k\cdot j\times i = k\cdot -k = -1$

Hence proved

Q:2 Find the volume of parallelopiped whose edges are represented by

$a = 3i + j - k$

$b = 2i - 3j + k$

$c = i - 3j - 4k$

Sol volume $= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 3 & 1 & -1 \\ 2 & -3 & 1 \\ 1 & -3 & -4 \end{vmatrix}$

$V = 3\begin{vmatrix} -3 & 1 \\ -3 & -4 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -3 \\ 1 & -3 \end{vmatrix}$

$= 3(12+3) - 1(-8-1) - 1(-6+3)$

$= 3(15) - 1(-9) - 1(-3)$

$= 45 + 9 + 3$

$= 57\ \text{unit}^3$

Q:3 For the vectors

$a = 3i + 2k$, $b = i + 2j + k$, $c = 0i - j + 4k$

verify that $a\cdot b\times c = b\cdot c\times a = c\cdot a\times b$ but $a\cdot b\times c = -c\cdot b\times a$

Sol

$a\cdot b\times c = \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} - 0 + 2\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}$

$= 3(8+1) - 0 + 2(-1-0)$

$= 27 - 2$

$a\cdot b\times c = 25 \longrightarrow$ (i)

$b\cdot c\times a = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 4 \\ 3 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix}$

$= 1(-2-0) - 2(0-12) + 1(0+3)$

$= -2 + 24 + 3 = 25$

Hence $b \cdot c \times a = 25 \longrightarrow (ii)$

Now $c \cdot a \times b = \begin{vmatrix} 0 & -1 & 4 \\ 3 & 0 & 2 \\ 1 & 2 & 1 \end{vmatrix}$

$= 0\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix}$

$= 0 + 1(3-2) + 4(6-0)$

$= 0 + 1 + 24$

$\Rightarrow c \cdot a \times b = 25 \longrightarrow (iii)$

From eqns (i), (ii) and (iii) it is proved that

$\boxed{a \cdot b \times c = b \cdot c \times a = c \cdot a \times b.}$

Now $c \times b \cdot a = c \cdot b \times a = \begin{vmatrix} 0 & -1 & 4 \\ 1 & 2 & 1 \\ 3 & 0 & 2 \end{vmatrix}$

$= 0\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$

$= 0 + 1(2-3) + 4(0-6)$

$= 0 + 1(-1) + 4(-6)$

$= -1-24$

$\boxed{c \times b \cdot a = -25}$

$c \times b \cdot a = -(a \cdot b \times c)$ Hence verified

Q:4 verify that the triple product of $i-j$, $j-k$, $k-i$ is zero.

Sol

$a = i-j = i-j+0k$

$b = j-k = 0i+j-k$

$c = k-i = -i+0j+k$

Now $a \cdot b \times c = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix}$

$= 1\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} + 0$

$= 1(1-0) + 1(0-1)$

$= 1(1) + 1(-1)$

$= 1-1$

$\boxed{a \cdot b \times c = 0}$ Hence proved.

Q:5 Find the value of c so that the vectors $ci+j+k$, $i+cj+k$, $i+j+ck$ are coplanar.

Sol

$V = 0 \quad \because$ vectors are coplanar

$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

$\Rightarrow \quad c\begin{vmatrix} c & 1 \\ 1 & c \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & c \end{vmatrix} + 1\begin{vmatrix} 1 & c \\ 1 & 1 \end{vmatrix} = 0$

$\Rightarrow \quad c(c^2-1) - 1(c-1) + 1(1-c) = 0$

$\Rightarrow \quad c^3 - c - c + 1 + 1 - c = 0$

$\Rightarrow \quad c^3 - 3c + 2 = 0$

$\Rightarrow \quad c^3 + 0c^2 - 3c + 2 = 0$

By synthetic division

Try 1

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

Hence 1 is root $\Rightarrow \boxed{c=1}$

The depressed eqn is

$1c^2 + 1c - 2 = 0$

$c^2 + 2c - c - 2 = 0$

$c(c+2) - 1(c+2) = 0$

$\Rightarrow (c+2)(c-1) = 0$

$c+2=0$ or $c-1=0$

$\boxed{c=-2}$ $\quad \boxed{c=1}$

Hence S.Set $= \{1, -2\}$ Ans

Q:6 Let $a = a_1 i + a_2 j + a_3 k$

$b = b_1 i + b_2 j + b_3 k$

Find $a \times b$ and prove that

Sol

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\Rightarrow a \times b = i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\boxed{a \times b = i(a_2b_3 - a_3b_2) - j(a_1b_3 - a_3b_1) + k(a_1b_2 - a_2b_1)}$$

(i) $a \times b$ is orthognal ($\perp$ar) to both a and b.

$\Rightarrow (a\times b)\cdot a = 0$

$(a\times b)\cdot a = \{(a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k\} \cdot (a_1 i + a_2 j + a_3 k)$

$\Rightarrow (a\times b)\cdot a = a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1)$

$= a_1a_2b_3 - a_1a_3b_2 - a_1a_2b_3 + a_2a_3b_1 + a_1a_3b_2 - a_2a_3b_1$

$= 0$

$\Rightarrow (a\times b)$ is orthognal to a.

Now $(a\times b)\cdot b = b_1(a_2b_3 - a_3b_2) - b_2(a_1b_3 - a_3b_1) + b_3(a_1b_2 - a_2b_1)$

$= a_2b_1b_3 - a_3b_1b_2 - a_1b_2b_3 + a_3b_1b_2 + a_1b_2b_3 - a_2b_1b_3$

$= 0$

$\Rightarrow (a\times b)$ is orthognal to b.

101

(ii) Find $|a\times b|^2$

Sol As $a\times b = (a_2b_3-a_3b_2)i - (a_1b_3-a_3b_1)j + (a_1b_2-a_2b_1)k$

$$\Rightarrow |a\times b| = \sqrt{(a_2b_3-a_3b_2)^2 + ((-1)(a_1b_3-a_3b_1))^2 + (a_1b_2-a_2b_1)^2}$$

taking square

$$\Rightarrow |a\times b|^2 = (a_2b_3-a_3b_2)^2 + (a_1b_3-a_3b_1)^2 + (a_1b_2-a_2b_1)^2$$

(iii) Find $|a\cdot b|^2$, $|a|^2$, $|b|^2$

Sol $(a\cdot b) = (a_1i+a_2j+a_3k)(b_1i+b_2j+b_3k)$

$\Rightarrow a\cdot b = a_1b_1 + a_2b_2 + a_3b_3$

$\Rightarrow |a\cdot b|^2 = (a_1b_1+a_2b_2+a_3b_3)^2$ Ans

$|a|^2 = ?$

As $a = a_1i + a_2j + a_3k$

$|a| = \sqrt{a_1^2+a_2^2+a_3^2}$

square

$|a|^2 = a_1^2 + a_2^2 + a_3^2$

$|b|^2 = ?$ As $b = b_1i + b_2j + b_3k$

$|b| = \sqrt{b_1^2+b_2^2+b_3^2}$

$\Rightarrow |b|^2 = b_1^2 + b_2^2 + b_3^3$

(iv) Show that $|a\times b|^2 = (a.a)(b.b) - (a.b)^2$

Sol As
L.H.S $|a\times b|^2 = (a_2b_3-a_3b_2)^2 + (a_1b_3-a_3b_1)^2 + (a_1b_2-a_2b_1)^2$

$= a_2^2b_3^2 + a_3^2b_2^2 - 2a_2a_3b_2b_3 + a_1^2b_3^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3$
$+ a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2$

$= a_1^2b_3^2 + a_3^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_2^2 - 2a_1a_3b_1b_3$
$- 2a_2a_3b_2b_3 - 2a_1a_2b_1b_2 \longrightarrow$ (i)

R.H.S

$a.a = (a_1i+a_2j+a_3k)(a_1i+a_2j+a_3k)$
$= a_1^2 + a_2^2 + a_3^2$

$b.b = (b_1i+b_2j+b_3k).(b_1i+b_2j+b_3k)$
$= b_1^2 + b_2^2 + b_3^2$

Now $(a.b)^2 = (a_1b_1 + a_2b_2 + a_3b_3)^2$

Then $(a.a)(b.b) - (a.b)^2$

$= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2$

$= a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2$
$-(a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2a_1b_1a_2b_2 + 2a_2b_2a_3b_3 + 2a_3b_3a_1b_1)$

$= \cancel{a_1^2b_1^2} + a_1^2b_2^2 + a_1^2b_3^2 + \cancel{a_2^2b_1^2} + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + \cancel{a_3^2b_3^2}$
$-\cancel{a_1^2b_1^2} - \cancel{a_2^2b_1^2} - \cancel{a_3^2b_3^2} - 2a_1a_2b_1b_2 - 2a_2a_3b_2b_3 - 2a_1a_3b_1b_3$

$$= a_1^2 b_3^2 + a_3^2 b_1^2 + a_1^2 b_2^2 + a_2^2 b_1^2 + a_2^2 b_3^2 + a_3^2 b_2^2$$
$$- 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3 - 2a_1a_2b_1b_2 \longrightarrow (2)$$

From eqns (1) and (2), it is proved

$$|a\times b|^2 = (a.a)(b.b) - (a.b)^2$$

Q:7 Do the points $(4,-2,1)$, $(5,1,6)$, $(2,2,-5)$ and $(3,5,0)$ lie in a plane.

Sol

let $A=(4,-2,1)$

$B=(5,1,6)$

$C=(2,2,-5)$

$D=(3,5,0)$

$$\vec{a} = \overrightarrow{AB} = \vec{B}-\vec{A} = (5,1,6)-(4,-2,1) = (1,3,5)$$
$$\vec{b} = \overrightarrow{AC} = \vec{C}-\vec{A} = (2,2,-5)-(4,-2,1) = (-2,4,-6)$$
$$\vec{c} = \overrightarrow{AD} = \vec{D}-\vec{A} = (3,5,0)-(4,-2,1) = (-1,7,-1)$$

Now $V = a.b\times c$

$$= \begin{vmatrix} 1 & 3 & 5 \\ -2 & 4 & -6 \\ -1 & 7 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 4 & -6 \\ 7 & -1 \end{vmatrix} - 3\begin{vmatrix} -2 & -6 \\ -1 & -1 \end{vmatrix} + 5\begin{vmatrix} -2 & 4 \\ -1 & 7 \end{vmatrix}$$

$$= 1(-4+42) - 3(2-6) + 5(-14+4)$$

$$= 38 + 12 - 50$$

$$= 0$$

Since $V=0 \Rightarrow$ The vectors are coplanar

Q:8 For what values of c the following vectors are coplanar?

Ch-03
P-18

(i) $u = i+2j+3k = (1,2,3)$

$v = 2i-3j+4k = (2,-3,4)$

$w = 3i+j+ck = (3,1,c)$

If u, v and w are coplanar, then

$u.v\times w = 0$

$$\Rightarrow \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 4 \\ 3 & 1 & c \end{vmatrix} = 0$$

$$\Rightarrow 1\begin{vmatrix} -3 & 4 \\ 1 & c \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 3 & c \end{vmatrix} + 3\begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 1(-3c-4) - 2(2c-12) + 3(2+9) = 0$$

$$\Rightarrow -3c-4-4c+24+33 = 0$$

$$\Rightarrow -7c+53 = 0 \Rightarrow 7c = 53 \Rightarrow \boxed{c = 53/7}$$

(ii) $u = (1,1,-1)$, $v = (1,-2,1)$, $w = (c,1,-c)$

for coplanar vectors

$u.v\times w = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ c & 1 & -c \end{vmatrix} = 0$$

$$\Rightarrow 1(2c-1) - 1(-c-c) - 1(1+2c) = 0$$

$$\Rightarrow 2c - 1 + 2c - 1 - 2c = 0$$

$$2c = 2 \Rightarrow \boxed{c = 1}$$

(iii) $u = (1,1,2)$ $\quad v = (2,3,1)$, $\quad w = (c,2,6)$

Sol For coplanar vectors

$$u \cdot v \times w = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ c & 2 & 6 \end{vmatrix} = 0$$

$$\Rightarrow 1\begin{vmatrix} 3 & 1 \\ 2 & 6 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ c & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ c & 2 \end{vmatrix} = 0$$

$$\Rightarrow 1(18-2) - 1(12-c) + 2(4-3c) = 0$$

$$\Rightarrow 16 - 12 + c + 8 - 6c = 0$$

$$\Rightarrow 12 - 5c = 0 \Rightarrow \boxed{c = 12/5}$$

106

Q:19 Find the volume of tetrahedron with the following.

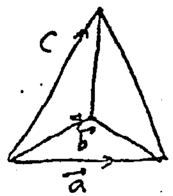
(a) vectors as coterminous edges

(a) $a = i+2j+3k$, $\quad b = 4i+5j+6k$, $\quad c = 0i+7j+8k$

$\Rightarrow a = (1,2,3)$, $\quad b = (4,5,6)$, $\quad c = (0,7,8)$

Sol volume of tetrahedron $= \frac{1}{6}(a \cdot b \times c)$

$$V = \frac{1}{6}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 7 & 8 \end{vmatrix}$$



$$\Rightarrow V = \frac{1}{6}\left\{1\begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 0 & 8 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 0 & 7 \end{vmatrix}\right\}$$

$$V = \frac{1}{6}\{1(40-42) - 2(32-0) + 3(28-0)\}$$

$$V = \frac{1}{6}\{(-2) - 64 + 84\} \Rightarrow V = \frac{1}{6}(-66+84) = \frac{1}{6}(18) = 3$$

Hence $\boxed{V = 3 \text{ unit}^3}$

(b) Points $A(2,3,1)$, $B(-1,-2,0)$, $C(0,2,-5)$ and $D(0,1,-2)$ as vertices.

Sol let $\vec{a} = \vec{AB} = \vec{B} - \vec{A} = (-1,-2,0) - (2,3,1) = (-3,-5,-1)$

$\vec{b} = \vec{AC} = \vec{C} - \vec{A} = (0,2,-5) - (2,3,1) = (-2,-1,-6)$

$\vec{c} = \vec{AD} = \vec{D} - \vec{A} = (0,1,-2) - (2,3,1) = (-2,-2,-3)$

Then $V = \frac{1}{6}(a \cdot b \times c)$

$$= \frac{1}{6}\begin{vmatrix} -3 & -5 & -1 \\ -2 & -1 & -6 \\ -2 & -2 & -3 \end{vmatrix}$$

$$= \frac{1}{6}\left\{+(-3)\begin{vmatrix} -1 & -6 \\ -2 & -3 \end{vmatrix} - (-5)\begin{vmatrix} -2 & -6 \\ -2 & -3 \end{vmatrix} + (-1)\begin{vmatrix} -2 & -1 \\ -2 & -2 \end{vmatrix}\right\}$$

$$= \frac{1}{6}\{(-3)(3-12) + 5(6-12) - 1(4-2)\}$$

$$= \frac{1}{6}\{(-3)(-9) + 5(-6) - 1(2)\}$$

$$= \frac{1}{6}\{27 - 30 - 2\}$$

$$= \frac{1}{6}\{-5\} = -5/6$$

$\Rightarrow \boxed{V = \frac{5}{6} \text{ unit}^3}$ $\quad \because$ volume can't be negative

End of chapter # 03