Exercise # 2.1

Q:1 Write the following products of matrices as a single matrix.

Sol (i) $\begin{bmatrix} 2 & -1 \\ 3 & -4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Sol $\begin{bmatrix} 2 & -1 \\ 3 & -4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (2\times1)+(-1\times0) & (2\times0)+(-1\times1) \\ (3\times1)+(-4\times0) & (3\times0)+(-4\times1) \\ (-1\times1)+(3\times0) & (-1\times0)+(3\times1) \end{bmatrix}$

$= \begin{bmatrix} 2+0 & 0-1 \\ 3+0 & 0-4 \\ -1+0 & 0+3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -4 \\ -1 & 3 \end{bmatrix}$ Ans

(ii) $\begin{bmatrix} 1 & 2 & -3 \\ -2 & -1 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0+0-3 & 0+2-0 & 1+0-0 \\ 0-0+4 & 0-1+0 & -2+0+0 \end{bmatrix}$

$= \begin{bmatrix} -3 & 2 & 1 \\ 4 & -1 & -2 \end{bmatrix}$ Ans

(iii) $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\left(\begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -5 \\ 3 & 5 \end{bmatrix}\right)$

$= \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2-1 & 3-5 \\ -4+3 & 1+5 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 6 \end{bmatrix}$

$= \begin{bmatrix} 1+2 & -2-12 \\ -1-1 & 2+6 \end{bmatrix} = \begin{bmatrix} 3 & -14 \\ -2 & 8 \end{bmatrix}$ Ans

Q:2 If possible, find matrix A

(i) $A\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 8 & -7 \end{bmatrix} \longrightarrow (i)$

Method # 01

Sol Let $\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} = B$ and $\begin{bmatrix} -2 & 5 \\ 8 & -7 \end{bmatrix} = C$

Then (i) $\Rightarrow AB = C$

$\Rightarrow A = CB^{-1} \longrightarrow (a)$

Now $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \Rightarrow |B| = (2\times1)-(0\times-3)$
$= 2-0$
$= 2$

and $B\ adj = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

As $B^{-1} = \frac{1}{|B|}[B\ adj]$

$\Rightarrow B^{-1} = \frac{1}{2}\begin{bmatrix}1 & 3\\ 0 & 2\end{bmatrix}$

P.T.V in eqn (i)

$A = CB^{-1}$

$\Rightarrow A = \begin{bmatrix}-2 & 5\\ 8 & -7\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & 3\\ 0 & 2\end{bmatrix}$

$\Rightarrow A = \frac{1}{2}\begin{bmatrix}-2 & 5\\ 8 & -7\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 2\end{bmatrix}$

$\Rightarrow A = \frac{1}{2}\begin{bmatrix}-2+0 & -6+10\\ 8-0 & 24-14\end{bmatrix}$

$\Rightarrow A = \frac{1}{2}\begin{bmatrix}-2 & 4\\ 8 & 10\end{bmatrix} \Rightarrow A = \begin{bmatrix}-2/2 & 4/2\\ 8/2 & 10/2\end{bmatrix}$

$\Rightarrow A = \begin{bmatrix}-1 & 2\\ 4 & 5\end{bmatrix}$ Ans

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Method # 02 $A = ?$

$A\begin{bmatrix}2 & -3\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-2 & 5\\ 8 & -7\end{bmatrix}$

Let $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ be the required matrix

then $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}2 & -3\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-2 & 5\\ 8 & -7\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2a+0 & -3a+b\\ 2c+0 & -3c+d\end{bmatrix} = \begin{bmatrix}-2 & 5\\ 8 & -7\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2a & -3a+b\\ 2c & -3c+d\end{bmatrix} = \begin{bmatrix}-2 & 5\\ 8 & -7\end{bmatrix}$

compare the corresponding elements, we get

$\Rightarrow 2a = -2$, $\quad -3a+b = 5$, $\quad -3c+d = -7$

$\Rightarrow \boxed{a = -1}$ $\quad \Rightarrow -3(-1)+b = 5$ $\quad \Rightarrow -3(4)+d = -7$

$3+b = 5$ $\quad \Rightarrow -12+d = -7$

& $2c = 8$ $\quad \Rightarrow \boxed{b = 2}$ $\quad \boxed{d = 5}$

$\Rightarrow \boxed{c = 4}$

Hence $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}-1 & 2\\ 4 & 5\end{bmatrix}$ Ans

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(ii) $A\begin{bmatrix}2 & -3\\ 4 & -6\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Sol

Let $\begin{bmatrix}2 & -3\\ 4 & -6\end{bmatrix} = B$

& $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I$

Then $AB = I$

$\Rightarrow A = IB^{-1} \longrightarrow$ (i)

Now $B = \begin{bmatrix}2 & -3\\ 4 & -6\end{bmatrix}$

$\Rightarrow |B| = -12-(-12) = 0$

Since $|B| = 0 \Rightarrow B^{-1}$ does not exist

$\Rightarrow$ From (i) A does not exist.

(iii) $A\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Sol let $\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix} = B$ and $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I$

Then $AB = I$

$\Rightarrow A = IB^{-1} \longrightarrow (i)$

Now $B = \begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix} \Rightarrow |B| = 4-(-6) = 10$

and $B\ adj = \begin{bmatrix}4 & 2\\ -3 & 1\end{bmatrix}$

$\therefore B^{-1} = \frac{1}{|B|}[B\ adj]$

$= \frac{1}{10}\begin{bmatrix}4 & 2\\ -3 & 1\end{bmatrix}$

Eqn (i) $\Rightarrow A = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\frac{1}{10}\begin{bmatrix}4 & 2\\ -3 & 1\end{bmatrix}$

$A = \frac{1}{10}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}4 & 2\\ -3 & 1\end{bmatrix}$

$A = \frac{1}{10}\begin{bmatrix}4-0 & 2+0\\ 0-3 & 0+1\end{bmatrix}$

$A = \frac{1}{10}\begin{bmatrix}4 & 2\\ -3 & 1\end{bmatrix} \Rightarrow A = \begin{bmatrix}4/10 & 2/10\\ -3/10 & 1/10\end{bmatrix}$

$\Rightarrow A = \begin{bmatrix}2/5 & 1/5\\ -3/10 & 1/10\end{bmatrix}$ Ans

Q:3 Solve the following eqns for x & y

(i) $\begin{bmatrix}x-2y\\ -x+2y\end{bmatrix} = \begin{bmatrix}2\\ 3\end{bmatrix}$

Sol compare the corresponding elements

$\Rightarrow x-2y = 2 \longrightarrow (i)$

$-x+2y = 3 \longrightarrow (ii)$

Add Eqn (i) & (ii), we get

$0+0=5$

$0=5$ which is not possible.

Hence the given system has no solution

(ii) $\begin{bmatrix}-3 & x & 2\\ 4y & 0 & 3\end{bmatrix} = \begin{bmatrix}x & -3 & 2\\ 0 & y & 3\end{bmatrix}$

Sol compare the elements

$\Rightarrow -3 = x$ & $0 = y$

$\Rightarrow$ S. Set $= \{-3, 0\}$ Ans

(iii) $\begin{bmatrix}4x+2 & 3 & 1\\ 5 & 3y+2 & -1\end{bmatrix} = \begin{bmatrix}2x & 3 & 1\\ 5 & 2y+4 & -1\end{bmatrix}$

Sol By comparing, we get

$4x+2 = 2x$ & $3y+2 = 2y+4$

$\Rightarrow 2x = -2$ $\Rightarrow y = 2$

$x = -1$

Hence S. Set $= \{x, y\} = \{-1, 2\}$ Ans

Q:4 If $A=\begin{bmatrix}1 & a\\ -1 & b\end{bmatrix}$ & $A^2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Find a & b?

Sol As $A^2 = A\cdot A$

$$\Rightarrow A^2 = \begin{bmatrix}1 & a\\ -1 & b\end{bmatrix}\begin{bmatrix}1 & a\\ -1 & b\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1-a & a+ab\\ -1-b & -a+b^2\end{bmatrix}$$

By comparison, we get

$1 = 1-a$, $0 = a+ab$

$\Rightarrow \boxed{a=0}$ & $0 = 0+0b \Rightarrow 0=b$

But $a=0$ & $b=0$ do not give $A^2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

So compare a_{21} elements

$0 = -1-b$

$\Rightarrow \boxed{b=-1}$

So $a=0$ and $b=-1$ give $A^2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Hence $\left.\begin{matrix}a=0\\ b=-1\end{matrix}\right\}$ Ans

Q:5 $A=\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}$ $B=\begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}$

Show that

(i) $(A+B)^2 \neq A^2+2AB+B^2$

Sol L.H.S $A+B = \begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix} + \begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}$

$$\Rightarrow A+B = \begin{bmatrix}3 & 5\\ -1 & 0\end{bmatrix}$$

Then $(A+B)^2 = (A+B)\cdot(A+B)$

$$= \begin{bmatrix}3 & 5\\ -1 & 0\end{bmatrix}\begin{bmatrix}3 & 5\\ -1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}9-0 & 15+0\\ -3-0 & -5+0\end{bmatrix}$$

$$\Rightarrow (A+B)^2 = \begin{bmatrix}9 & 15\\ -3 & -5\end{bmatrix} \longrightarrow (i)$$

R.H.S $A^2 = A\cdot A$

$$A^2 = \begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix}1+0 & 2-2\\ 0-0 & 0+1\end{bmatrix} \Rightarrow A^2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$2AB = 2\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}$$

$$= 2\begin{bmatrix}2-2 & 3+2\\ 0+1 & 0-1\end{bmatrix}$$

$\Rightarrow 2AB = 2\begin{bmatrix}0 & 5\\ 1 & -1\end{bmatrix} \Rightarrow 2AB = \begin{bmatrix}0 & 10\\ 2 & -2\end{bmatrix}$

and $B^2 = B \cdot B$

$B^2 = \begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}$

$= \begin{bmatrix}4-3 & 6+3\\ -2-1 & -3+1\end{bmatrix}$

$B^2 = \begin{bmatrix}1 & 9\\ -3 & -2\end{bmatrix}$

Finally $A^2 + 2AB + B^2$

$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}0 & 10\\ 2 & -2\end{bmatrix} + \begin{bmatrix}1 & 9\\ -3 & -2\end{bmatrix}$

$= \begin{bmatrix}2 & 19\\ -1 & -3\end{bmatrix} \longrightarrow (ii)$

From eqns (i) and (ii), it is proved

$(A+B)^2 \neq A^2 + 2AB + B^2.$

(ii) $(A-B)^2 \neq A^2 - 2AB + B^2$

L.H.S $A-B = \begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix} - \begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}$

$\Rightarrow A-B = \begin{bmatrix}-1 & -1\\ 1 & -2\end{bmatrix}$

Now $(A-B)^2 = (A-B)\cdot(A-B)$

$= \begin{bmatrix}-1 & -1\\ 1 & -2\end{bmatrix} \cdot \begin{bmatrix}-1 & -1\\ 1 & -2\end{bmatrix}$

$= \begin{bmatrix}1-1 & 1+2\\ -1-2 & -1+4\end{bmatrix}$

$= \begin{bmatrix}0 & 3\\ -3 & 3\end{bmatrix} \longrightarrow (i)$

R.H.S $A^2 - 2AB + B^2$

$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 10\\ 2 & -2\end{bmatrix} + \begin{bmatrix}1 & 9\\ -3 & -2\end{bmatrix}$

$= \begin{bmatrix}1 & -10\\ -2 & 3\end{bmatrix} + \begin{bmatrix}1 & 9\\ -3 & -2\end{bmatrix}$

$= \begin{bmatrix}2 & -1\\ -5 & 1\end{bmatrix} \longrightarrow (ii)$

From eqns (i) and (ii), it is proved that

$(A-B)^2 \neq A^2 - 2AB + B^2.$

(iii) $(A+B)(A-B) \neq (A^2 - B^2)$

L.H.S

$$A+B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow A+B = \begin{bmatrix} 3 & 5 \\ -1 & 0 \end{bmatrix}$$

and

$$A-B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$$

Then

$$(A+B)(A-B) = \begin{bmatrix} 3 & 5 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$$

$$\Rightarrow (A+B)(A-B) = \begin{bmatrix} -3+5 & -3-10 \\ 1+0 & 1-0 \end{bmatrix}$$

$$\Rightarrow (A+B)(A-B) = \begin{bmatrix} 2 & -13 \\ 1 & 1 \end{bmatrix} \longrightarrow (i)$$

R.H.S

As $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $B^3 = \begin{bmatrix} 1 & 9 \\ -3 & -2 \end{bmatrix}$

So

$$A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 9 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -9 \\ 3 & 2 \end{bmatrix} \longrightarrow (ii)$$

From eqns (i) and (ii), it is proved $(A+B)(A-B) \neq A^2 - B^2$

Q:6

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix}$$

Show that $(A+B)^t = A^t + B^t$

L.H.S

$$A+B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix}$$

$$\Rightarrow A+B = \begin{bmatrix} 3 & -1 & 2 & 3 \\ 4 & 4 & 1 & 9 \\ 3 & -1 & 3 & 5 \end{bmatrix}$$

take transpose, we get

$$\Rightarrow (A+B)^t = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix} \longrightarrow (i)$$

R.H.S

$$A^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} \quad \& \quad B^t = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix}$$

$$A^t + B^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix}$$

$$\Rightarrow A^t + B^t = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix} \longrightarrow (ii)$$

From eqns (i) and (ii) it is proved

$$(A+B)^t = A^t + B^t$$

Q:7 Find the inverse

(i) $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

Sol let $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

$\Rightarrow |A| = (2\times2) - (3\times-1)$

$\Rightarrow |A| = 4+3 \Rightarrow |A| = 7$

and $A\ adj = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

Then $A^{-1} = \frac{1}{|A|}[A\ adj]$

$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$ Ans

(ii) $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

let $B = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \Rightarrow |B| = 1-0 \Rightarrow |B| = 1$

and $B\ adj = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}$

Then $B^{-1} = \frac{1}{|B|}[B\ adj]$

$= \frac{1}{1}\begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}$ Ans

(iii) $\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}$

let $P = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \Rightarrow |P| = 1-0 \Rightarrow |P| = 1$

and $P\ adj = \begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix}$

Then $P^{-1} = \frac{1}{|P|}[P\ adj]$

$= \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix}$ Ans

Q:8 Solve the following systems of linear eqns.

(i) $x - y = 2 \longrightarrow (i)$

$2x + y = 3 \longrightarrow (ii)$

Eqn (i) + Eqn (ii)

$\Rightarrow 3x = 5$

$\Rightarrow x = 5/3$

Now $x - y = 2$

$\Rightarrow \frac{5}{3} - y = 2$

$\Rightarrow \frac{5}{3} - 2 = y$

$\Rightarrow \frac{-1}{3} = y$

Hence S.set $= \{x, y\}$

$= \{5/3, -1/3\}$ Ans

S. set $= \{5/3, -1/3\}$ Ans

Method # 02 : By Matrices

$$x - y = 2$$
$$2x + y = 3$$

Sol: In matrix form

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

let $\Downarrow$ A, $\Downarrow$ X, $\Downarrow$ B

Then $AX = B$

$$\Rightarrow X = A^{-1}B \longrightarrow (i)$$

As $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \Rightarrow |A| = 1-(-2)$

$$\Rightarrow |A| = 3$$

and $A\ adj = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

Then $A^{-1} = \frac{1}{|A|}\left[A\ adj\right]$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

Eqn (i) $\Rightarrow$ $X = A^{-1}B$

$$\Rightarrow X = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{3} \begin{bmatrix} 2+3 \\ -4+3 \end{bmatrix}$$

$$X = \frac{1}{3} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5/3 \\ -1/3 \end{bmatrix}$$

Hence S.set $= \left\{\frac{5}{3}, \frac{-1}{3}\right\}$

Quote: Education is a progressive discovery of our ignorance. (Will Durant 1885 – 1981)

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(ii) $x - y = 3 \longrightarrow$ (i)

$x + y = 5 \longrightarrow$ (ii)

Method # 01

Eqn (i) + Eqn (ii), we get

$$\Rightarrow 2x = 8$$

$$\Rightarrow \boxed{x = 4}$$

Now Eqn (i) $\Rightarrow x - y = 3$

$$\Rightarrow 4 - y = 3$$

$$\Rightarrow \boxed{1 = y}$$

S.set $= \{x, y\}$

$= \{4, 1\}$ Ans

Method # 02

$$x - y = 3$$
$$x + y = 5$$

In matrix form

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

let $\Downarrow$ A, $\Downarrow$ X, $\Downarrow$ B

$$AX = B$$

$$\Rightarrow X = A^{-1}B \rightarrow (i)$$

$$|A| = 1-(-1) = 2$$

$$A\ adj = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

then $A^{-1} = \frac{1}{|A|}\left[A\ adj\right]$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Eqn (i) $\Rightarrow$ $X = A^{-1}B$

$$\Rightarrow X = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{2} \begin{bmatrix} 3+5 \\ -3+5 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{2} \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$\Rightarrow$ S.set $= \{4, 1\}$ Ans

Engr. MAJID AMIN
MS. Mechanical Engineering
UET Peshawar.
Mob 0333-9470690

(iii) $4x-3y=-1 \rightarrow (i)$

$2x+5y=19 \rightarrow (ii)$

Method #01

Xing eqn (ii) by 2 and then subtract from eqn (i), we get

$\Rightarrow 4x-3y=-1$

$\Rightarrow \underline{\ \ 4x+10y=38\ \ }$

$\ \ \ \ -\ \ \ -\ \ \ \ -$

$-13y=-39$

$\boxed{y=3}$

Now Eqn (i) $4x-3y=-1$

$\Rightarrow 4x-3(3)=-1$

$\Rightarrow 4x-9=-1$

$\Rightarrow 4x=8$

$\Rightarrow \boxed{x=2}$

Hence S.set $=\{2,3\}$ Ans

Method #02

In matrix form

$$\begin{bmatrix}4 & -3\\ 2 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-1\\ 19\end{bmatrix}$$

let $\Downarrow A \quad \Downarrow X \quad \Downarrow B$

$AX=B$

$\Rightarrow X=A^{-1}B \longrightarrow (i)$

$|A|=20-(-6)=26$

$A.adj=\begin{bmatrix}5 & 3\\ -2 & 4\end{bmatrix}$

$A^{-1}=\frac{1}{|A|}[A\,adj]$

$A^{-1}=\frac{1}{26}\begin{bmatrix}5 & 3\\ -2 & 4\end{bmatrix}$

Eqn (i) $\Rightarrow X=A^{-1}B$

$X=\frac{1}{26}\begin{bmatrix}5 & 3\\ -2 & 4\end{bmatrix}\begin{bmatrix}-1\\ 19\end{bmatrix}$

$X=\frac{1}{26}\begin{bmatrix}-5+57\\ 2+76\end{bmatrix}$

$\Downarrow$

$\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{26}\begin{bmatrix}52\\ 78\end{bmatrix}$

$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2\\ 3\end{bmatrix}$

S.Set $=\{2,3\}$ Ans

Exercise 2.2

Q:1 Write the following sums as a single matrix

(i) $\begin{bmatrix}4\\ 5\\ 6\end{bmatrix}+\begin{bmatrix}3\\ -2\\ -1\end{bmatrix}$

$=\begin{bmatrix}7\\ 3\\ 5\end{bmatrix}$ Ans

(ii) $\begin{bmatrix}-2 & 3 & 4\end{bmatrix}+\begin{bmatrix}1 & 0 & -3\end{bmatrix}$

$=\begin{bmatrix}-1 & 3 & 1\end{bmatrix}$ Ans

(iii) $\begin{bmatrix}-2 & 3 & 2\\ 1 & 2 & 0\end{bmatrix}+\begin{bmatrix}3 & 1 & 1\\ 0 & -1 & -2\end{bmatrix}$

$=\begin{bmatrix}1 & 4 & 3\\ 1 & 1 & -2\end{bmatrix}$ Ans

(iv) $\begin{bmatrix}1 & 0 & -1\\ 2 & 3 & 4\\ 0 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 3 & 4\\ 2 & 3 & 5\\ 0 & 1 & 4\end{bmatrix}$

$=\begin{bmatrix}2 & 3 & 3\\ 4 & 6 & 9\\ 0 & 2 & 6\end{bmatrix}$ Ans

Q:2 Write the following product as a single matrix.

(i) $-2\begin{bmatrix}-1 & 2 & 4\\ -3 & 5 & -2\end{bmatrix}$

$=\begin{bmatrix}2 & -4 & -8\\ 6 & -10 & 4\end{bmatrix}$ Ans

(ii) $\begin{bmatrix}-1 & -2 & -3\end{bmatrix}\begin{bmatrix}3\\ -2\\ 1\end{bmatrix}$

$=\begin{bmatrix}(-1\times3)+(-2\times-2)+(-3\times1)\end{bmatrix}$

$=\begin{bmatrix}-3+4-3\end{bmatrix}=\begin{bmatrix}-2\end{bmatrix}$ Ans

(iii) $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$

Sol

$= \begin{bmatrix} 6-6+0 & 0+3+1 \\ 0+4+0 & 0-2+1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 4 \\ 4 & -1 \end{bmatrix}$ Ans

(iv) $\frac{1}{3}\begin{bmatrix} 6 & 3 & -9 \\ 12 & -15 & 3 \\ 0 & 6 & 9 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{3}(6) & \frac{1}{3}(3) & \frac{1}{3}(-9) \\ \frac{1}{3}(12) & \frac{1}{3}(-15) & \frac{1}{3}(3) \\ \frac{1}{3}(0) & \frac{1}{3}(6) & \frac{1}{3}(9) \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & -3 \\ 4 & -5 & 1 \\ 0 & 2 & 3 \end{bmatrix}$ Ans

32 (v) $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 3 \\ 4 & 2 & -1 \end{bmatrix}$

Sol

$= \begin{bmatrix} 0+3+4 & 2+6+2 & 0-9-1 \\ 0-2+4 & 0-4+2 & 0+6-1 \\ 0-0+4 & 1-0+2 & 0+0-1 \end{bmatrix}$

$= \begin{bmatrix} 7 & 10 & -10 \\ 2 & -2 & 5 \\ 4 & 3 & -1 \end{bmatrix}$ Ans

Q:3 let $A = \begin{bmatrix} 2 & -5 & 1 \\ 3 & 0 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & -2 \\ 0 & -1 & -1 \end{bmatrix}$

Find $2A + 3B - 4C$

Sol $2A + 3B - 4C = 2\begin{bmatrix} 2 & -5 & 1 \\ 3 & 0 & -4 \end{bmatrix} + 3\begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & 5 \end{bmatrix} - 4\begin{bmatrix} 0 & 1 & -2 \\ 0 & -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -10 & 2 \\ 6 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 3 & -6 & -9 \\ 0 & -3 & 15 \end{bmatrix} - \begin{bmatrix} 0 & 4 & -8 \\ 0 & -4 & -4 \end{bmatrix}$

$= \begin{bmatrix} 7 & -16 & -7 \\ 6 & -3 & 7 \end{bmatrix} - \begin{bmatrix} 0 & 4 & -8 \\ 0 & -4 & -4 \end{bmatrix}$

$= \begin{bmatrix} 7 & -20 & 1 \\ 6 & 1 & 11 \end{bmatrix}$ Ans

Q:4 $A = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 4 \\ 4 & 4 & 1 \end{bmatrix}$ ∴ show that $\frac{1}{3}A^2 - 2A - 9I = 0$

Sol $A^2 = A \cdot A = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 4 \\ 4 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 4 \\ 4 & 4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+16+16 & 4+4+16 & 4+16+4 \\ 4+4+16 & 16+1+16 & 16+4+4 \\ 4+16+4 & 16+4+4 & 16+16+1 \end{bmatrix}$

$= \begin{bmatrix} 33 & 24 & 24 \\ 24 & 33 & 24 \\ 24 & 24 & 33 \end{bmatrix}$

L.H.S $\frac{1}{3}A^2 - 2A - 9I$

$= \frac{1}{3}\begin{bmatrix} 33 & 24 & 24 \\ 24 & 33 & 24 \\ 24 & 24 & 33 \end{bmatrix} - 2\begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 4 \\ 4 & 4 & 1 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 11 & 8 & 8 \\ 8 & 11 & 8 \\ 8 & 8 & 11 \end{bmatrix} - \begin{bmatrix} 2 & 8 & 8 \\ 8 & 2 & 8 \\ 8 & 8 & 2 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O = R.H.S$$

Q:5 $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ $B = \begin{bmatrix} 0 & i^2 \\ -i^2 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

verify the following

(i) $A^2 = B^2 = C^2 = -I$

Sol

$A^2 = A.A$

$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$= \begin{bmatrix} i^2+0 & 0-0 \\ 0-0 & 0+i^2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= -1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= -I \longrightarrow (i)$$

and $B^2 = B.B$

$$\Rightarrow B^2 = \begin{bmatrix} 0 & i^2 \\ -i^2 & 0 \end{bmatrix}\begin{bmatrix} 0 & i^2 \\ -i^2 & 0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 0-1 & 0-0 \\ 0+0 & -1+0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= -I \longrightarrow (ii)$$

(ii) $AB = -BA = -C$

Sol

$$AB = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & i^2 \\ -i^2 & 0 \end{bmatrix}$$

$i^2 = -1$
$-i^2 = -(-1)$
$= 1$

$$AB = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 0+0 & -i+0 \\ 0-i & -0-0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \longrightarrow (i)$$

Now

$$-BA = -\begin{bmatrix} 0 & i^2 \\ -i^2 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$= -\begin{bmatrix} 0-0 & 0+i \\ i+0 & 0-0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \longrightarrow (ii)$$

Finally

$$-C = -\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \longrightarrow (iii)$$

From equs (i), (ii) and (iii), it is proved that $AB = -BA = -C$.

(iii) $BC = -CB = -A$

First BC

$$BC = \begin{bmatrix} 0 & i^2 \\ -i^2 & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0-i & 0-0 \\ 0+0 & i+0 \end{bmatrix}$$

$$BC = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \longrightarrow (a)$$

34 Now $-CB$

$$-CB = -\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & i^2 \\ -i^2 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0+i & -0+0 \\ 0+0 & -i+0 \end{bmatrix}$$

$$= -\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \longrightarrow (b)$$

Finally

$$-A = -\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \longrightarrow (c)$$

From (a), (b) and (c), it is proved that

$$BC = -CB = -A$$

(ii) $CA = -AC = -B$

First

$$CA = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 0-i^2 \\ i^2+0 & 0-0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longrightarrow (i)$$

Now

$$-AC = -\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0+0 & i^2+0 \\ 0-i^2 & 0-0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longrightarrow (ii)$$

and

$$-B = -\begin{bmatrix} 0 & i^2 \\ -i^2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longrightarrow (iii)$$

From (i), (ii) and (iii), it is proved that

$$CA = -AC = -B$$

Q:6

$$A = \begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 1 & 1 \\ 4 & 3 & 2 \\ 0 & 1 & -5 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 & 6 \\ 4 & 5 & 0 \\ 1 & 3 & 4 \end{bmatrix}$$

verify that

(i) $A+B = B+A$

L.H.S

$$A+B = \begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 1 \\ 4 & 3 & 2 \\ 0 & 1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 1 & 5 \\ 2 & 9 & 3 \\ 3 & 3 & -4 \end{bmatrix} \longrightarrow (i)$$

R.H.S

$$B+A=\begin{bmatrix}6&1&1\\4&3&2\\0&1&-5\end{bmatrix}+\begin{bmatrix}5&0&4\\-2&6&1\\3&2&1\end{bmatrix}$$

$$=\begin{bmatrix}11&1&5\\2&9&3\\3&3&-4\end{bmatrix}\longrightarrow(ii)$$

From (i) and (ii) it is verified that

$$A+B=B+A.$$

(ii) $A+(B+C)=(A+B)+C$

L.H.S

$$B+C=\begin{bmatrix}6&1&1\\4&3&2\\0&1&-5\end{bmatrix}+\begin{bmatrix}3&2&6\\4&5&0\\1&3&4\end{bmatrix}$$

$$=\begin{bmatrix}9&3&7\\8&8&2\\1&4&-1\end{bmatrix}$$

And $A+(B+C)=\begin{bmatrix}5&0&4\\-2&6&1\\3&2&1\end{bmatrix}+\begin{bmatrix}9&3&7\\8&8&2\\1&4&-1\end{bmatrix}$

$$=\begin{bmatrix}14&3&11\\6&14&3\\4&6&0\end{bmatrix}\longrightarrow(i)$$

R.H.S

$$A+B=\begin{bmatrix}5&0&4\\-2&6&1\\3&2&1\end{bmatrix}+\begin{bmatrix}6&1&1\\4&3&2\\0&1&-5\end{bmatrix}$$

$$=\begin{bmatrix}11&1&5\\2&9&3\\3&3&-4\end{bmatrix}$$

Then $(A+B)+C=\begin{bmatrix}11&1&5\\2&9&3\\3&3&-4\end{bmatrix}+\begin{bmatrix}3&2&6\\4&5&0\\1&3&4\end{bmatrix}=\begin{bmatrix}14&3&11\\6&14&3\\4&6&0\end{bmatrix}\rightarrow(ii)$

From (i) & (ii), it is verified that $A+(B+C)=(A+B)+C$

(iii) $A+O=O+A=A$

First $A+O$

$$A+O=\begin{bmatrix}5&0&4\\-2&6&1\\3&2&1\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$=\begin{bmatrix}5&0&4\\-2&6&1\\3&2&1\end{bmatrix}=A$$

Now

$$O+A=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}5&0&4\\-2&6&1\\3&2&1\end{bmatrix}$$

$$=\begin{bmatrix}5&0&4\\-2&6&1\\3&2&1\end{bmatrix}=A$$

Hence verified

(iv) $A+(-A)=(-A)+A=O$

$A=\begin{bmatrix}5&0&4\\-2&6&1\\3&2&1\end{bmatrix}$ then $-A=-\begin{bmatrix}5&0&4\\-2&6&1\\3&2&1\end{bmatrix}$

$$\Rightarrow -A=\begin{bmatrix}-5&0&-4\\2&-6&-1\\-3&-2&-1\end{bmatrix}$$

Then $A+(-A)=\begin{bmatrix}5&0&4\\-2&6&1\\3&2&1\end{bmatrix}+\begin{bmatrix}-5&0&-4\\2&-6&-1\\-3&-2&-1\end{bmatrix}$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\longrightarrow(i)$$

$$=O$$

Now $(-A)+A = \begin{bmatrix} -5 & 0 & -4 \\ 2 & -6 & -1 \\ -3 & -2 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \longrightarrow$ (ii)

Hence verified

(v) $(ab)A = a(bA)$

L.H.S $(ab)A$

$= (ab)\begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5ab & 0ab & 4ab \\ -2ab & 6ab & 1ab \\ 3ab & 2ab & 1ab \end{bmatrix} = a\begin{bmatrix} 5b & 0b & 4b \\ -2b & 6b & b \\ 3b & 2b & b \end{bmatrix}$

$= a(bA) =$ R.H.S

(vi) $a(A+B) = aA + aB$

L.H.S $a(A+B)$

$= a\left\{ \begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 1 \\ 4 & 3 & 2 \\ 0 & 1 & -5 \end{bmatrix} \right\}$

$= a\left\{ \begin{bmatrix} 11 & 1 & 5 \\ 2 & 9 & 3 \\ 3 & 3 & -4 \end{bmatrix} \right\}$

$= \begin{bmatrix} 11a & 1a & 5a \\ 2a & 9a & 3a \\ 3a & 3a & -4a \end{bmatrix} \longrightarrow$ (i)

R.H.S $aA + aB$

$= a\begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix} + a\begin{bmatrix} 6 & 1 & 1 \\ 4 & 3 & 2 \\ 0 & 1 & -5 \end{bmatrix}$

$= \begin{bmatrix} 5a & 0 & 4a \\ -2a & 6a & 1a \\ 3a & 2a & 1a \end{bmatrix} + \begin{bmatrix} 6a & 1a & 1a \\ 4a & 3a & 2a \\ 0a & 1a & -5a \end{bmatrix}$

$= \begin{bmatrix} 11a & 1a & 5a \\ 2a & 7a & 3a \\ 3a & 3a & -4a \end{bmatrix} \longrightarrow$ (ii)

From (i) and (ii), it is verified that

$a(A+B) = aA + aB.$

(vii) $(a+b)A = aA + bA$

L.H.S $(a+b)A$

$= (a+b)\begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} (a+b)5 & (a+b)0 & (a+b)4 \\ (a+b)(-2) & (a+b)6 & (a+b)1 \\ (a+b)3 & (a+b)2 & (a+b)1 \end{bmatrix}$

$= \begin{bmatrix} 5a+5b & 0 & 4a+4b \\ -2a-2b & 6a+6b & a+b \\ 3a+3b & 2a+2b & a+b \end{bmatrix} \longrightarrow$ (i)

R.H.S $\quad aA + bA$

$$= a\begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix} + b\begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5a & 0 & 4a \\ -2a & 6a & 1a \\ 3a & 2a & 1a \end{bmatrix} + \begin{bmatrix} 5b & 0 & 4b \\ -2b & 6b & 1b \\ 3b & 2b & b \end{bmatrix}$$

$$= \begin{bmatrix} 5a+5b & 0 & 4a+4b \\ -2a-2b & 6a+6b & 1a+1b \\ 3a+3b & 2a+2b & a+b \end{bmatrix} \longrightarrow (ii)$$

From (i) & (ii) it is proved

$$(a+b)A = aA + bA.$$

(viii) $A(BC) = (AB)C$

L.H.S

$$BC = \begin{bmatrix} 6 & 1 & 1 \\ 4 & 3 & 2 \\ 0 & 1 & -5 \end{bmatrix}\begin{bmatrix} 3 & 2 & 6 \\ 4 & 5 & 0 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 18+4+1 & 12+5+3 & 36+0+4 \\ 12+12+2 & 8+15+6 & 24+0+8 \\ 0+4-5 & 0+5-15 & 0+0-20 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 20 & 40 \\ 26 & 29 & 32 \\ -1 & -10 & -20 \end{bmatrix}$$

Now

$$A(BC) = \begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 23 & 20 & 40 \\ 26 & 29 & 32 \\ -1 & -10 & -20 \end{bmatrix}$$

$$\Rightarrow A(BC) = \begin{bmatrix} 115+0-4 & 100+0-40 & 200+0-80 \\ -46+156-1 & -40+174-10 & -80+192-20 \\ 69+52-1 & 60+58-10 & 120+64-20 \end{bmatrix}$$

$$= \begin{bmatrix} 111 & 60 & 120 \\ 109 & 124 & 92 \\ 120 & 108 & 164 \end{bmatrix} \longrightarrow (i)$$

R.H.S

$$AB = \begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 6 & 1 & 1 \\ 4 & 3 & 2 \\ 0 & 1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 30+0+0 & 5+0+4 & 5+0-20 \\ -12+24+0 & -2+18+1 & -2+12-5 \\ 18+8+0 & 3+6+1 & 3+4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 9 & -15 \\ 12 & 17 & 5 \\ 26 & 10 & 2 \end{bmatrix}$$

Now

$$(AB)C = \begin{bmatrix} 30 & 9 & -15 \\ 12 & 17 & 5 \\ 26 & 10 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 & 6 \\ 4 & 5 & 0 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 90+36-15 & 60+45-45 & 180+0-60 \\ 36+68+5 & 24+85+15 & 72+0+20 \\ 78+40+2 & 52+50+6 & 156+0+8 \end{bmatrix}$$

$$= \begin{bmatrix} 111 & 60 & 120 \\ 109 & 124 & 92 \\ 120 & 108 & 164 \end{bmatrix} \longrightarrow (ii)$$

From (i) & (ii), it is verified that $A(BC) = (AB)C$

(ix) $A(B+C) = AB + AC$

L.H.S

$$B+C = \begin{bmatrix} 6 & 1 & 1 \\ 4 & 3 & 2 \\ 0 & 1 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 6 \\ 4 & 5 & 0 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 & 7 \\ 8 & 8 & 2 \\ 1 & 4 & -1 \end{bmatrix}$$

Now $$A(B+C) = \begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 9 & 3 & 7 \\ 8 & 8 & 2 \\ 1 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 45+0+4 & 15+0+16 & 35+0-4 \\ -18+48+1 & -6+48+4 & -14+12-1 \\ 27+16+1 & 9+16+4 & 21+4-1 \end{bmatrix}$$

$$= \begin{bmatrix} 49 & 31 & 31 \\ 31 & 46 & -3 \\ 44 & 29 & 24 \end{bmatrix} \longrightarrow (i)$$

R.H.S

$$AB = \begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 6 & 1 & 1 \\ 4 & 3 & 2 \\ 0 & 1 & -5 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 30+0+0 & 5+0+4 & 5+0-20 \\ -12+24+0 & -2+18+1 & -2+12-5 \\ 18+8+0 & 3+6+1 & 3+4-5 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 30 & 9 & -15 \\ 12 & 17 & 5 \\ 26 & 10 & 2 \end{bmatrix}$$

Now $$AC = \begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 6 \\ 4 & 5 & 0 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 15+0+4 & 10+0+12 & 30+0+16 \\ -6+24+1 & -4+30+3 & -12+0+4 \\ 9+8+1 & 6+10+3 & 18+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 22 & 46 \\ 19 & 29 & -8 \\ 18 & 19 & 22 \end{bmatrix}$$

Now $$AB + AC = \begin{bmatrix} 30 & 9 & -15 \\ 12 & 17 & 5 \\ 26 & 10 & 2 \end{bmatrix} + \begin{bmatrix} 19 & 22 & 46 \\ 19 & 29 & -8 \\ 18 & 19 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 49 & 31 & 31 \\ 31 & 46 & -3 \\ 44 & 29 & 24 \end{bmatrix} \longrightarrow (ii)$$

From (i) and (ii), it is verified that
$A(B+C) = AB + AC$

(x) $(A+B)C = AC + BC$

L.H.S

$$A+B = \begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 1 \\ 4 & 3 & 2 \\ 0 & 1 & -5 \end{bmatrix}$$

$$\Rightarrow A+B = \begin{bmatrix} 11 & 1 & 5 \\ 2 & 9 & 3 \\ 3 & 3 & -4 \end{bmatrix}$$

Now $(A+B)C = \begin{bmatrix} 11 & 1 & 5 \\ 2 & 9 & 3 \\ 3 & 3 & -4 \end{bmatrix}\begin{bmatrix} 3 & 2 & 6 \\ 4 & 5 & 0 \\ 1 & 3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 33+4+5 & 22+5+15 & 66+0+20 \\ 6+36+3 & 4+45+9 & 12+0+12 \\ 9+12-4 & 6+15-12 & 18+0-16 \end{bmatrix}$$

$$= \begin{bmatrix} 42 & 42 & 86 \\ 45 & 58 & 24 \\ 17 & 9 & 2 \end{bmatrix} \longrightarrow (i)$$

<u>R.H.S</u>

$$AC = \begin{bmatrix} 5 & 0 & 4 \\ -2 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 6 \\ 4 & 5 & 0 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\Rightarrow AC = \begin{bmatrix} 15+0+4 & 10+0+12 & 30+0+16 \\ -6+24+1 & -4+30+3 & -12+0+4 \\ 9+8+1 & 6+10+3 & 18+0+4 \end{bmatrix}$$

$$\Rightarrow AC = \begin{bmatrix} 19 & 22 & 46 \\ 19 & 29 & -8 \\ 18 & 19 & 22 \end{bmatrix}$$

Now $BC = \begin{bmatrix} 6 & 1 & 1 \\ 4 & 3 & 2 \\ 0 & 1 & -5 \end{bmatrix}\begin{bmatrix} 3 & 2 & 6 \\ 4 & 5 & 0 \\ 1 & 3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 18+4+1 & 12+5+3 & 36+0+4 \\ 12+12+2 & 8+15+6 & 24+0+8 \\ 0+4-5 & 0+5-15 & 0+0-20 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 20 & 40 \\ 26 & 29 & 32 \\ -1 & -10 & -20 \end{bmatrix}$$

Now $AC + BC = \begin{bmatrix} 19 & 22 & 46 \\ 19 & 29 & -8 \\ 18 & 19 & 22 \end{bmatrix} + \begin{bmatrix} 23 & 20 & 40 \\ 26 & 29 & 32 \\ -1 & -10 & -20 \end{bmatrix}$

$$= \begin{bmatrix} 42 & 42 & 86 \\ 45 & 58 & 24 \\ 17 & 9 & 2 \end{bmatrix} \longrightarrow (ii)$$

From (i) and (ii) it is verified that

$$(A+B)C = AC + BC$$

<u>Q:7</u> Determine whether commutative property w.r.t multiplication holds in each of the following cases or not?

(i) $A = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$ $B = \begin{bmatrix} s & t \\ -t & s \end{bmatrix}$

<u>Finding AB</u>

$$AB = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}\begin{bmatrix} s & t \\ -t & s \end{bmatrix}$$

$$= \begin{bmatrix} ps-qt & pt+qs \\ -qs-pt & -qt+ps \end{bmatrix}$$

Now $BA = \begin{bmatrix} s & t \\ -t & s \end{bmatrix}\begin{bmatrix} p & q \\ -q & p \end{bmatrix}$

$$= \begin{bmatrix} ps-qt & qs+pt \\ -pt-qs & -qt+ps \end{bmatrix} = AB$$

Since $AB = BA \Rightarrow$ Commutative property w.r.t $\times$ holds

(ii) $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix}$

Sol

$$AB = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+3+4 & 6+1+6 \\ 2+3+6 & 4+1+9 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 13 \\ 11 & 14 \end{bmatrix} \longrightarrow (i)$$

Now $BA = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3+4 & 1+2 & 2+6 \\ 9+2 & 3+1 & 6+3 \\ 6+6 & 2+3 & 4+9 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 3 & 8 \\ 11 & 4 & 9 \\ 12 & 5 & 13 \end{bmatrix} \longrightarrow (ii)$$

From (i) and (ii), it is verified that

$$AB \neq BA$$

$\Rightarrow$ commutative property w.r.t multiplication does not hold.

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Q:8 $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$ show that

(i) $(A^t)^t = A$ (ii) $AA^t \neq A^t A$

Sol $(A^t)^t = A$

L.H.S $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$

take transpose

$\Rightarrow A^t = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix}$

Again take transpose

$(A^t)^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$

$= A = R.H.S$

L.H.S

$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix}$

$AA^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+0 & 3-2+0 \\ 3-2+0 & 9+1+16 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 \\ 1 & 26 \end{bmatrix}$$

Now $A^t . A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 10 & -1 & 12 \\ -1 & 5 & -4 \\ 12 & -4 & 16 \end{bmatrix}$$

Hence $AA^t \neq A A^t$

Q:9 Solve for X.

(i) $X - 3A = 2B$ where $A = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 4 \end{bmatrix}$

$\Rightarrow X = 2B + 3A$

$$= 2\begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 4 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 2 \\ 6 & -2 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ -6 & 6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 & 11 \\ 0 & 4 & 11 \end{bmatrix} \text{ Ans}$$

(ii) $2(X-A)=B$ where $A=\begin{bmatrix}1&2&2\\3&-1&2\end{bmatrix}$ & $B=\begin{bmatrix}4&6&2\\0&-4&2\end{bmatrix}$

$\Rightarrow X-A=\frac{1}{2}B$

$\Rightarrow X=\frac{1}{2}B+A$

$\Rightarrow X=\frac{1}{2}\begin{bmatrix}4&6&2\\0&-4&2\end{bmatrix}+\begin{bmatrix}1&2&2\\3&-1&2\end{bmatrix}$

$\Rightarrow X=\begin{bmatrix}2&3&1\\0&-2&1\end{bmatrix}+\begin{bmatrix}1&2&2\\3&-1&2\end{bmatrix}$

$\Rightarrow X=\begin{bmatrix}3&5&3\\3&-3&3\end{bmatrix}$ Ans

Quote:

Do not worry about your difficulties in mathematics. I can assure you mine are still greater.

(Albert Einstein)

Exercise # 2.3

Q:1 $A=\begin{bmatrix}1&3&1\\-1&2&0\\2&0&-2\end{bmatrix}$ Find $A_{11}, A_{21}, A_{23}, A_{33}$ & $|A|$.

Sol

$A_{11}=+\begin{vmatrix}2&0\\0&-2\end{vmatrix}=+(-4-0)=-4$

$A_{21}=-\begin{vmatrix}3&1\\0&-2\end{vmatrix}=-(-6-0)=6$

$A_{23}=-\begin{vmatrix}1&3\\2&0\end{vmatrix}=-(0-6)=6$

$A_{33}=+\begin{vmatrix}1&3\\-1&2\end{vmatrix}=+(2+3)=6$

To find $|A|$:

Expand by R_1

$|A|=+1\begin{vmatrix}2&0\\0&-2\end{vmatrix}-3\begin{vmatrix}-1&0\\2&-2\end{vmatrix}+1\begin{vmatrix}-1&2\\2&0\end{vmatrix}$

$=+1(-4-0)-3(2-0)+1(0-4)$

$=-4-6-4$

$=-14$ Ans

Q:2 Without evaluating state the reason for the following equilities.

(i) $\begin{vmatrix}1&2&0\\3&1&0\\-1&2&0\end{vmatrix}$ (ii) $\begin{vmatrix}1&2&3\\-8&4&-12\\2&-1&3\end{vmatrix}=0$

$\because C_3=0$

If -4 is taken common from R_2 then $R_2=R_3$

(iii) $\begin{vmatrix} 1 & 3 & -2 \\ 3 & -1 & 1 \\ 2 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ -2 & 1 & 4 \end{vmatrix}$

$\because |A^t| = |A|$

(iv) $\begin{vmatrix} 3 & 2 & 0 \\ 1 & 1 & -3 \\ 2 & 4 & -6 \end{vmatrix} = -3 \begin{vmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{vmatrix}$

$\because -3$ common from C_3

(v) $\begin{vmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 1 & -1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$

$\because$ when two rows of a det are interchanged then det of original is (-1) multiplied by det of the new. & $(R_2 \longleftrightarrow R_3)$

(vi) $\begin{vmatrix} 2 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 5 & 5 & 6 \\ 1 & 2 & 2 \end{vmatrix}$

$\because R_2 + 2R_3$

Q:3 Let A is a square matrix of order 3 then verify that $|A^t| = |A|$

Sol Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

R.H.S

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\Rightarrow |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23})$
$+ a_{13}(a_{21}a_{32} - a_{31}a_{22})$

$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23}$
$+ a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$ (i)

L.H.S $A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$\Rightarrow |A^t| = a_{11} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix}$

$\Rightarrow |A^t| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32})$
$+ a_{31}(a_{12}a_{23} - a_{13}a_{22})$

$\Rightarrow |A^t| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32}$
$+ a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}$

Rearranging, we get

$\Rightarrow |A^t| = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{31}a_{23}$
$- a_{13}a_{31}a_{22}$ (ii)

From (i) and (ii), it is proved
$|A^t| = |A|$

Q:4 Evaluate the following determinants

(i) $\begin{vmatrix} 0 & 1 & 3 \\ -1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix}$

Expand by R_1

$= 0\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}$

$= 0 - 1(-1-2) + 3(-1-4)$

$= -1(-3) + 3(-5)$

$= 3 - 15 = -12$ Ans

(ii) $\begin{vmatrix} 3 & 4 & -2 \\ 2 & 4 & -6 \\ -4 & 2 & 0 \end{vmatrix}$

$= 3\begin{vmatrix} 4 & -6 \\ 2 & 0 \end{vmatrix} - 4\begin{vmatrix} 2 & -6 \\ -4 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 4 \\ -4 & 2 \end{vmatrix}$

$= 3(0+12) - 4(0-24) - 2(4+16)$

$= 3(12) - 4(-24) - 2(20)$

$= 36 + 96 - 40$

$= 92$ Ans

(iii) $\begin{vmatrix} 3 & 1 & 2 \\ 6 & -5 & 4 \\ -9 & 8 & -7 \end{vmatrix}$

Expand by C_1

$= 3\begin{vmatrix} -5 & 4 \\ 8 & -7 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 8 & -7 \end{vmatrix} + 2\begin{vmatrix} 6 & -5 \\ -9 & 8 \end{vmatrix}$

$= 3(35-32) - 6(-7-16) + 2(48-45)$

$= 3(3) - 6(-23) + 2(3)$

$= 9 + 138 + 6 = 153$ Ans

(iv) $\begin{vmatrix} 2 & 1 & -3 \\ 1 & 1 & 0 \\ -2 & 3 & 4 \end{vmatrix}$

$= 2\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ -2 & 4 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix}$

$= 2(4-0) - 1(4-0) - 3(3+2)$

$= 2(4) - 1(4) - 3(5)$

$= 8 - 4 - 15$

$= -11$ Ans

Q:5 Show that

(i) $\begin{vmatrix} a & b & c \\ l & m & n \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & l & x \\ b & m & y \\ c & n & z \end{vmatrix}$

L.H.S $\begin{vmatrix} a & b & c \\ l & m & n \\ x & y & z \end{vmatrix}$

take transpose, we get

$= \begin{vmatrix} a & l & x \\ b & m & y \\ c & n & z \end{vmatrix} = R.H.S$

(ii) $\begin{vmatrix} a & b & c \\ 1-3a & 2-3b & 3-3c \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$

L.H.S : $\begin{vmatrix} a & b & c \\ 1-3a & 2-3b & 3-3c \\ 4 & 5 & 6 \end{vmatrix}$

By property, we have

$$= \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} a & b & c \\ -3a & -3b & -3c \\ 4 & 5 & 6 \end{vmatrix}$$ -3 is common in R_2

$$= \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} + (-3)\begin{vmatrix} a & b & c \\ a & b & c \\ 4 & 5 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} - 3\ (0) \quad \because R_1 = R_2$$

$$= \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = R.H.S$$

(iii) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix} = 0$$

L.H.S $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$$

$R_3 + R_2$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$$

a+b+c common from R_3

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (a+b+c)(0) \quad \because R_1 = R_3$$

$$= 0 = R.H.S$$

(iv) $$\begin{vmatrix} bc & ca & ab \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

L.H.S

$$\begin{vmatrix} bc & ac & ab \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{abc}{a} & \frac{abc}{b} & \frac{abc}{c} \\ \frac{a^2}{a} & \frac{b^2}{b} & \frac{c^2}{c} \\ \frac{a^3}{a} & \frac{b^3}{b} & \frac{c^3}{c} \end{vmatrix}$$

$\frac{1}{a}$ common from C_1

$\frac{1}{b}$ " " C_2

$\frac{1}{c}$ " " C_3

$$= \frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}\begin{vmatrix} abc & abc & abc \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Now abc common from R_1

$$= \frac{1}{abc}\ abc\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = R.H.S$$

(v) $\begin{vmatrix} bc & a^3 & \frac{1}{a} \\ ac & b^3 & \frac{1}{b} \\ ab & c^3 & \frac{1}{c} \end{vmatrix} = 0$

L.H.S $\begin{vmatrix} bc & a^3 & \frac{1}{a} \\ ac & b^3 & \frac{1}{b} \\ ab & c^3 & \frac{1}{c} \end{vmatrix}$

$= \begin{vmatrix} \frac{abc}{a} & a^3 & 1/a \\ \frac{abc}{b} & b^3 & 1/b \\ \frac{abc}{c} & c^3 & 1/c \end{vmatrix}$

abc common from C_1

$= abc \begin{vmatrix} 1/a & a^3 & 1/a \\ 1/b & b^3 & 1/b \\ 1/c & c^3 & 1/c \end{vmatrix}$

$= abc \; (0) \quad \because C_1 = C_3$

$= 0 = R.H.S$

(vi) $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

L.H.S $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}$

$R_2 - R_1, \; R_3 - R_1$

$= \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & \beta-\alpha & \beta^2-\alpha^2 \\ 0 & \gamma-\alpha & \gamma^2-\alpha^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & \beta-\alpha & (\beta+\alpha)(\beta-\alpha) \\ 0 & \gamma-\alpha & (\gamma+\alpha)(\gamma-\alpha) \end{vmatrix}$

$\beta-\alpha$ common from R_2
$\gamma-\alpha$ " " R_3

$= (\beta-\alpha)(\gamma-\alpha) \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & 1 & \beta+\alpha \\ 0 & 1 & \gamma+\alpha \end{vmatrix}$

Now $R_3 - R_2$

$$= (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & 1 & \beta+\alpha \\ 0 & 0 & \gamma-\beta \end{vmatrix}$$

$(\gamma+\alpha)-(\beta+\alpha)$
$=\gamma+\alpha-\beta-\alpha$
$=\gamma-\beta$

Expand by R_3

$$= (\beta-\alpha)(\gamma-\alpha)\left\{ 0 - 0 + (\gamma-\beta)\begin{vmatrix} 1 & \alpha \\ 0 & 1 \end{vmatrix} \right\}$$

$$= (\beta-\alpha)(\gamma-\alpha)\ \{(\gamma-\beta)(1-0)\}$$

$$= (\beta-\alpha)(\gamma-\alpha)\ (\gamma-\beta)$$

$$= (-1)(-\beta+\alpha)\quad(\gamma-\alpha)\quad(-1)(-\gamma+\beta)$$

$+$

$$= +\ (\alpha-\beta)\ (\gamma-\alpha)(\beta-\gamma)$$

$= R.H.S$

(viii) $$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\ (\alpha+\beta+\gamma)$$

L.H.S $$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix}$$

$C_2 - C_1,\quad C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^3 & \beta^3-\alpha^3 & \gamma^3-\alpha^3 \end{vmatrix}$$

$a^3-b^3=(a-b)(a^2+b^2+ab)$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^3 & (\beta-\alpha)(\beta^2+\alpha^2+\alpha\beta) & (\gamma-\alpha)(\gamma^2+\alpha^2+\gamma\alpha) \end{vmatrix}$$

$\beta-\alpha$ common from C_2
$\gamma-\alpha$ 〃 〃 C_3

$$= \begin{vmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \alpha^3 & \beta^2+\alpha^2+\alpha\beta & \gamma^2+\alpha^2+\gamma\alpha \end{vmatrix}\ (\beta-\alpha)(\gamma-\alpha)$$

$C_3 - C_2$

$(\gamma^2+\alpha^2+\gamma\alpha)-(\beta^2+\alpha^2+\alpha\beta)$
$=\gamma^2+\alpha^2+\gamma\alpha-\beta^2-\alpha^2-\alpha\beta$

$$= (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \alpha^3 & \beta^2+\alpha^2+\alpha\beta & \gamma^2-\beta^2+\gamma\alpha-\alpha\beta \end{vmatrix}$$

Expand by C_3

$$= (\beta-\alpha)(\gamma-\alpha)\left\{ 0-0+(\gamma^2-\beta^2+\gamma\alpha-\alpha\beta)\begin{vmatrix} 1 & 0 \\ \alpha & 1 \end{vmatrix} \right\}$$

$$= (\beta-\alpha)(\gamma-\alpha)\ \{(\gamma^2-\beta^2+\gamma\alpha-\alpha\beta)(1-0)\}$$

$= (\beta-\alpha)(\gamma-\alpha)\left\{\gamma^2-\beta^2+\gamma\alpha-\alpha\beta\right\}$

$= (\beta-\alpha)(\gamma-\alpha)\left\{(\gamma+\beta)(\gamma-\beta) + \alpha(\gamma-\beta)\right\}$

take $\gamma-\beta$ as common

$= (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)\left\{\gamma+\beta+\alpha\right\}$

$= -1(-\beta+\alpha)\ (\gamma-\alpha)\ (-1)(-\gamma+\beta)\ (\alpha+\beta+\gamma)$

$= +\ (\alpha-\beta)(\gamma-\alpha)\ (\beta-\gamma)(\alpha+\beta+\gamma)$

$=$ R.H.S

(viii) $\begin{vmatrix} \gamma\cos\theta & 0 & \gamma\sin\theta \\ 1 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{vmatrix} = \gamma$

L.H.S $\begin{vmatrix} \gamma\cos\theta & 0 & \gamma\sin\theta \\ 1 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{vmatrix}$

Expand by R_1

$= \gamma\cos\theta\begin{vmatrix} 1 & 0 \\ 0 & \cos\theta \end{vmatrix} - 0 + \gamma\sin\theta\begin{vmatrix} 1 & 1 \\ -\sin\theta & 0 \end{vmatrix}$

$= \gamma\cos\theta(\cos\theta-0) + \gamma\sin\theta(0+\sin\theta)$

$= \gamma\cos^2\theta + \gamma\sin^2\theta$

$= \gamma(\cos^2\theta+\sin^2\theta) = \gamma(1) = \gamma =$ R.H.S

Q:6 Identify singular and non singular matrices.

(i) $A = \begin{bmatrix} 7 & 1 & 3 \\ 6 & 2 & -2 \\ 5 & 1 & 1 \end{bmatrix}$

Note If $|A| = 0$
$\Rightarrow A$ is singular matrix

Sol Finding $|A|$

$\Rightarrow |A| = \begin{vmatrix} 7 & 1 & 3 \\ 6 & 2 & -2 \\ 5 & 1 & 1 \end{vmatrix}$

Expand by R_1

$|A| = 7\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 6 & -2 \\ 5 & 1 \end{vmatrix} + 3\begin{vmatrix} 6 & 2 \\ 5 & 1 \end{vmatrix}$

$= 7(2+2) - 1(6+10) + 3(6-10)$

$= 7(4) - 1(16) + 3(-4)$

$= 28 - 16 - 12$

$\Rightarrow |A| = 28-28$

$\Rightarrow |A| = 0$

$\Rightarrow A$ is a singular matrix.

(ii) Let $B = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -2 & 1 \\ -2 & -3 & 2 \end{bmatrix}$

Sol

$\Rightarrow |B| = \begin{vmatrix} 1 & -1 & 1 \\ 3 & -2 & 1 \\ -2 & -3 & 2 \end{vmatrix}$

Expand by R_1

$\Rightarrow |B| = 1\begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ -2 & -3 \end{vmatrix}$

$\Rightarrow |B| = 1(-4+3) + 1(6+2) + 1(-9-4)$

$\Rightarrow |B| = 1(-1) + 1(8) + 1(-13)$

$|B| = -1 + 8 - 13$

$\Rightarrow |B| = -6$

Since $|B| \neq 0$

$\Rightarrow$ B is a non-singular matrix.

(iii) Let $C = \begin{bmatrix} 3 & 2 & -3 \\ 3 & 6 & -3 \\ -1 & 0 & 1 \end{bmatrix}$

Sol

$|C| = \begin{vmatrix} 3 & 2 & -3 \\ 3 & 6 & -3 \\ -1 & 0 & 1 \end{vmatrix}$

Expand by R_1

$\Rightarrow |C| = 3\begin{vmatrix} 6 & -3 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & -3 \\ -1 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 3 & 6 \\ -1 & 0 \end{vmatrix}$

$\Rightarrow |C| = 3(6-0) - 2(3-3) - 3(0+6)$

$|C| = 3(6) - 2(0) - 3(6)$

$|C| = 18 - 0 - 18$

$\Rightarrow |C| = 0$

Hence C is a singular matrix

Q.7 Find the value of λ if A is a singular matrix?

$A = \begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix}$

Sol

$|A| = +(-\lambda)\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & -\lambda \end{vmatrix} + 0$

$|A| = -\lambda(\lambda^2 - 1) - 1(-\lambda - 0)$

$\Rightarrow |A| = -\lambda^3 + \lambda + \lambda$

$\Rightarrow |A| = -\lambda^3 + 2\lambda$

$\Rightarrow |A| = 2\lambda - \lambda^3$

$\Rightarrow |A| = \lambda(2 - \lambda^2)$

Since A is singular (given)

$\Rightarrow |A| = 0$

$\Rightarrow \lambda(2 - \lambda^2) = 0$

$\lambda(2 - \lambda^2) = 0$

$\lambda = 0$ or $2 - \lambda^2 = 0$

$\lambda = 0$ $\quad 2 = \lambda^2$

$\Rightarrow \lambda = \pm\sqrt{2}$

Hence $\lambda = \{0, \pm\sqrt{2}\}$ Ans

Q:8 Solve for x

(i) $\begin{vmatrix} x & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 4 & 5 \end{vmatrix} = 9$

Sol Expand by C_1

$\Rightarrow +x\begin{vmatrix} -1 & 1 \\ 4 & 5 \end{vmatrix} - 0 + 0 = 9$

$\Rightarrow x(-5-4) = 9$

$\Rightarrow x(-9) = 9$

$\Rightarrow -9x = 9 \Rightarrow x = \frac{9}{-9} \Rightarrow \boxed{x=-1}$ Ans

(ii) $\begin{vmatrix} -1 & 0 & 1 \\ x^2 & 1 & x \\ 2 & 3 & 4 \end{vmatrix} = -6$

Sol Expand by R_1

$\Rightarrow +(-1)\begin{vmatrix} 1 & x \\ 3 & 4 \end{vmatrix} - 0 + 1\begin{vmatrix} x^2 & 1 \\ 2 & 3 \end{vmatrix} = -6$

$\Rightarrow (-1)(4-3x) + 1(3x^2-2) = -6$

$\Rightarrow -4+3x+3x^2-2 = -6$

$\Rightarrow 3x^2+3x-6 = -6$

$\Rightarrow 3x^2+3x-6+6=0$

$\Rightarrow 3x^2+3x=0$

$3x$ is common

$\Rightarrow 3x(x+1)=0$

$3x=0$ or $x+1=0$

$\boxed{x=0}$ or $\boxed{x=-1}$

Hence S.set $= \{0,-1\}$

Q:9 Show that the inverse of a matrix (if exists) is unique.

Sol Let A is a matrix and it has two inverses P and Q.

Then by definition

$AP = I \longrightarrow$ (i)

$AQ = I \longrightarrow$ (ii)

Compare eqn (i) and (ii), we get

$AP = AQ$

$\Rightarrow P=Q$

which shows that A can't have two diffrent inverses.

$\Rightarrow$ The inverse of a matrix (if exists) is unique.

Q:10 $A = \begin{bmatrix} 0 & 2 & 2 \\ -1 & 3 & 2 \\ 1 & 0 & 5 \end{bmatrix}$ Find A^{-1} ?

Sol As $A^{-1} = \frac{1}{|A|}[A\ adj] \longrightarrow (i)$

we have,

$$|A| = \begin{vmatrix} 0 & 2 & 2 \\ -1 & 3 & 2 \\ 1 & 0 & 5 \end{vmatrix}$$

$$\Rightarrow |A| = 0 - 2\begin{vmatrix} -1 & 2 \\ 1 & 5 \end{vmatrix} + 2\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix}$$

$$\Rightarrow |A| = -2(-5-2) + 2(0-3)$$

$$\Rightarrow |A| = -2(-7) + 2(-3)$$

$$\Rightarrow |A| = 14 - 6$$

$$\Rightarrow \boxed{|A| = 8}$$

And the cofactors are

$$A_{11} = +\begin{vmatrix} 3 & 2 \\ 0 & 5 \end{vmatrix} = +(15-0) = +15$$

$$A_{12} = -\begin{vmatrix} -1 & 2 \\ 1 & 5 \end{vmatrix} = -(-5-2) = +7$$

$$A_{13} = +\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} = +(0-3) = -3$$

$$A_{21} = -\begin{vmatrix} 2 & 2 \\ 0 & 5 \end{vmatrix} = -(10-0) = -10$$

$$A_{22} = +\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = +(0-2) = -2$$

$$A_{23} = -\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -(0-2) = 2$$

$$A_{31} = +\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = +(4-6) = -2$$

$$A_{32} = -\begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} = -(0+2) = -2$$

$$A_{33} = +\begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} = +(0+2) = 2$$

Then the adjoint matrix is

$$A\ adj = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$A\ adj = \begin{bmatrix} 15 & 7 & -3 \\ -10 & -2 & 2 \\ -2 & -2 & 2 \end{bmatrix}^t \Rightarrow A\ adj = \begin{bmatrix} 15 & -10 & -2 \\ 7 & -2 & -2 \\ -3 & 2 & 2 \end{bmatrix}$$

Eqn (i) $\Rightarrow$ $A^{-1} = \frac{1}{|A|}[A\ adj]$

$$A^{-1} = \frac{1}{8}\begin{bmatrix} 15 & -10 & -2 \\ 7 & -2 & -2 \\ -3 & 2 & 2 \end{bmatrix} \quad \text{Ans}$$

OR

$$A^{-1} = \begin{bmatrix} 15/8 & -5/4 & -1/4 \\ 7/8 & -1/4 & -1/4 \\ -3/8 & 1/4 & 1/4 \end{bmatrix} \quad \text{Ans}$$

Q:11 If $A = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$ show that $A^{-1} = \frac{1}{|A|}$

Sol

$|A| = (3\times 2) - (4\times -1)$

$\Rightarrow |A| = 6+4 \Rightarrow |A| = 10$

Now $A\ adj = \begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}$

Then $A^{-1} = \frac{1}{|A|}[A\ adj]$

$\Rightarrow A^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 8 \\ -4 & 3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} \\ -\frac{2}{5} & \frac{3}{10} \end{bmatrix}$

$\Rightarrow |A^{-1}| = \left(\frac{1}{5}\times\frac{3}{10}\right) - \left(\frac{-2}{5}\times\frac{1}{10}\right)$

$= \frac{3}{50} + \frac{2}{50}$

$= \frac{3+2}{50}$

$= \frac{5}{50}$

$|A^{-1}| = \frac{1}{10}$

$\Rightarrow |A^{-1}| = \frac{1}{|A|} \quad \because 10 = |A|$

Hence proved.

Q:12 Verify that $(AB)^{-1} = B^{-1}A^{-1} \longrightarrow ①$

CH-02
P-15

where $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$

Finding L.H.S of ①

$AB = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$

$\Rightarrow AB = \begin{bmatrix} -2+6 & 2+9 \\ -1+0 & 1+0 \end{bmatrix}$

$\Rightarrow AB = \begin{bmatrix} 4 & 11 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow |AB| = 4-(-11) \Rightarrow |AB| = 15$

Now $AB\ adj = \begin{bmatrix} 1 & -11 \\ 1 & 4 \end{bmatrix}$

Then $(AB)^{-1} = \frac{1}{|AB|}[AB\ adj]$

$\Rightarrow (AB)^{-1} = \frac{1}{15}\begin{bmatrix} 1 & -11 \\ 1 & 4 \end{bmatrix} \longrightarrow ⓐ$

From eqns ⓐ and ⓑ, we have

$(AB)^{-1} = B^{-1}A^{-1}$

Finding R.H.S

$|B| = (-1\times 3) - (2\times 1)$ & $|A| = (0\times 2) - (1\times 3)$

$\Rightarrow |B| = -3-2$ $\quad \Rightarrow |A| = 0-3$

$\Rightarrow |B| = -5$ $\quad |A| = -3$

$B\ adj = \begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix}$ and $A\ adj = \begin{bmatrix} 0 & -3 \\ -1 & 2 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|}[B\ adj]$ & $A^{-1} = \frac{1}{|A|}[A\ adj]$

$\Rightarrow B^{-1} = \frac{1}{-5}\begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix}$ $\quad A^{-1} = \frac{1}{-3}\begin{bmatrix} 0 & -3 \\ -1 & 2 \end{bmatrix}$

Finally $B^{-1}A^{-1}$

$= \frac{1}{-5}\begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix} \frac{1}{-3}\begin{bmatrix} 0 & -3 \\ -1 & 2 \end{bmatrix}$

$= \frac{1}{-5}\times\frac{1}{-3}\begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 0 & -3 \\ -1 & 2 \end{bmatrix}$

$= \frac{1}{15}\begin{bmatrix} 0+1 & -9-2 \\ -0+1 & 6-2 \end{bmatrix}$

$= \frac{1}{15}\begin{bmatrix} 1 & -11 \\ 1 & 4 \end{bmatrix} \longrightarrow ⓑ$

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Q:13 A and B are non-singular matrices show that

(i) $(A^{-1})^{-1} = A$

Sol As $A \cdot A^{-1} = I$

$\Rightarrow A^{-1}$ is inverse of A

Also $A^{-1} A = I$

$\Rightarrow A$ is inverse of A^{-1}

i.e $(A^{-1})^{-1} = A$

Note

$A \cdot A^{-1} = I$

If $A \cdot B = I$

$\Rightarrow B$ is inverse of A

or $A^{-1} = B$

(ii) $(AB)^{-1} = B^{-1} \cdot A^{-1}$

Sol As $(AB)(B^{-1}A^{-1})$

$= AB \cdot B^{-1}A^{-1}$

$= ABB^{-1}A^{-1}$

$= AIA^{-1}$

$= A \cdot A^{-1}$

$= I$

Since $(AB)(B^{-1}A^{-1}) = I$

$\Rightarrow B^{-1}A^{-1}$ is inverse of AB

$\Rightarrow (AB)^{-1} = B^{-1} A^{-1}$

Hence proved.

Q:14

verify that

$$(AB)^t = B^t A^t$$

where $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 0 \end{bmatrix}$

L.H.S

$$AB = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 0 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 2-2+9 & 4-2+0 \\ 1+0+3 & 2+0+0 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 9 & 2 \\ 4 & 2 \end{bmatrix}$$

Now $(AB)^t = \begin{bmatrix} 9 & 4 \\ 2 & 2 \end{bmatrix} \longrightarrow (i)$

R.H.S

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

$\Rightarrow A^t = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 1 \end{bmatrix}$ & $B^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 0 \end{bmatrix}$

Now $B^t A^t$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-2+9 & 1+0+3 \\ 4-2+0 & 2+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 4 \\ 2 & 2 \end{bmatrix} \longrightarrow (ii)$$

From (i) and (ii), it is proved

$$(AB)^t = B^t A^t$$

(ii) $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & -2 \end{bmatrix}$

L.H.S

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & -2 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 1+4+0 & 1+6-0 \\ -1+2+4 & -1+3-8 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 5 & 7 \\ 5 & -6 \end{bmatrix}$$

$$\Rightarrow (AB)^t = \begin{bmatrix} 5 & 5 \\ 7 & -6 \end{bmatrix}$$

R.H.S

$$A^t = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 0 & 4 \end{bmatrix} \quad B^t = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & -2 \end{bmatrix}$$

$$B^t . A^t = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 0 & 4 \end{bmatrix}$$

$$B^t A^t = \begin{bmatrix} 1+4+0 & -1+2+4 \\ 1+6-0 & -1+3-8 \end{bmatrix}$$

$$B^t A^t = \begin{bmatrix} 5 & 5 \\ 7 & -6 \end{bmatrix}$$

Hence $(AB)^t = B^t A^t$.

Q.15 $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$, verify that $(A^{-1})^t = (A^t)^{-1}$

L.H.S $|A| = 2-(-3)$

$$\Rightarrow |A| = 5$$

$$A\ adj = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} [A\ adj]$$

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/5 & -3/5 \\ 1/5 & 2/5 \end{bmatrix}$$

take transpose

$$(A^{-1})^t = \begin{bmatrix} 1/5 & 1/5 \\ -3/5 & 2/5 \end{bmatrix} \longrightarrow (i)$$

R.H.S $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

$$A^t = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$|A^t| = 2-(-3)$$

$$\Rightarrow |A^t| = 5$$

and $A^t\ adj = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

Now $(A^t)^{-1} = \frac{1}{|A^t|} (A^t adj)$

$$(A^t)^{-1} = \frac{1}{5} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$$

$$(A^t)^{-1} = \begin{bmatrix} 2/5 & 3/5 \\ -1/5 & 1/5 \end{bmatrix}$$

(ii)

From (i) and (ii), it is proved

$$(A^{-1})^t = (A^t)^{-1}.$$

Exercise # 2.4

Q:1 $A = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 & 7 \\ 6 & -8 & 3 \\ 7 & 3 & 1 \end{bmatrix}$

Show that A and B are symmetric. Also show that $(A+B)$ is symmetric.

Sol Finding A^t and B^t

$$A^t = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix} \qquad B^t = \begin{bmatrix} 5 & 6 & 7 \\ 6 & -8 & 3 \\ 7 & 3 & 1 \end{bmatrix}$$

$\Rightarrow A^t = A$ $\qquad \Rightarrow B^t = B$

$\Rightarrow A$ is symmetric. $\qquad \Rightarrow B$ is symmetric

Now $A+B$ is symmetric

$$A+B = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 6 & 7 \\ 6 & -8 & 3 \\ 7 & 3 & 1 \end{bmatrix}$$

NOTE
If $M^t = M$
$\Rightarrow M$ is symmetric

$$\Rightarrow A+B = \begin{bmatrix} 6 & 3 & 11 \\ 3 & -6 & -2 \\ 11 & -2 & 1 \end{bmatrix}$$

take transpose

$$(A+B)^t = \begin{bmatrix} 6 & 3 & 11 \\ 3 & -6 & -2 \\ 11 & -2 & 1 \end{bmatrix}$$

Since $(A+B)^t = (A+B)$

$\Rightarrow (A+B)$ is symmetric matrix

Q:2 $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -6 & 11 \\ 6 & 0 & -7 \\ -11 & 7 & 0 \end{bmatrix}$

Show that $A+B$ is skew symmetric.

Sol

$$A+B = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -6 & 11 \\ 6 & 0 & -7 \\ -11 & 7 & 0 \end{bmatrix}$$

$$\Rightarrow (A+B) = \begin{bmatrix} 0 & -5 & 9 \\ 5 & 0 & -4 \\ -9 & 4 & 0 \end{bmatrix}$$

Note
If $M^t = -M$
then M is skew symmetric

$$\Rightarrow (A+B)^t = \begin{bmatrix} 0 & 5 & -9 \\ -5 & 0 & 4 \\ 9 & -4 & 0 \end{bmatrix}$$

$$\Rightarrow (A+B)^t = -\begin{bmatrix} 0 & -5 & 9 \\ 5 & 0 & -4 \\ -9 & 4 & 0 \end{bmatrix}$$

$\Rightarrow (A+B)^t = -(A+B)$

$\Rightarrow (A+B)$ is skew symmetric.

Q:3 $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ -2 & 3 & 4 \end{bmatrix}$ Show that

(a) $A + A^t$ is symmetric

Sol $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ -2 & 3 & 4 \end{bmatrix}$ & $A^t = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 5 & 3 \\ 1 & 6 & 4 \end{bmatrix}$

Now $A + A^t = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ -2 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -2 \\ 2 & 5 & 3 \\ 1 & 6 & 4 \end{bmatrix}$

$\Rightarrow A + A^t = \begin{bmatrix} 6 & 6 & -1 \\ 6 & 10 & 9 \\ -1 & 9 & 8 \end{bmatrix}$

taking transpose

$\Rightarrow (A + A^t)^t = \begin{bmatrix} 6 & 6 & -1 \\ 6 & 10 & 9 \\ -1 & 9 & 8 \end{bmatrix}$

$(A + A^t)^t = A + A^t$

$\Rightarrow$ $A + A^t$ is symmetric matrix.

(b) $A - A^t$ is skew symmetric

Sol $A - A^t = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ -2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 4 & -2 \\ 2 & 5 & 3 \\ 1 & 6 & 4 \end{bmatrix}$

$\Rightarrow A - A^t = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 3 \\ -3 & -3 & 0 \end{bmatrix}$

take transpose

$\Rightarrow (A - A^t)^t = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -3 \\ 3 & 3 & 0 \end{bmatrix}$

$\Rightarrow (A - A^t)^t = - \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 3 \\ -3 & -3 & 0 \end{bmatrix}$

$\Rightarrow (A - A^t)^t = - \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 3 \\ -3 & -3 & 0 \end{bmatrix}$

$\Rightarrow (A - A^t)^t = -(A - A^t)$

$\Rightarrow$ $A - A^t$ is skew symmetric matrix.

Q.4 If A is a square matrix of order 3, then show that

(a) $(A + A^t)$ is symmetric

Sol As $(A + A^t)^t = A^t + (A^t)^t$

$= A^t + A$

$\Rightarrow (A + A^t)^t = A + A^t$

$\Rightarrow$ $A + A^t$ is symmetric matrix

(b) $A - A^t$ is skew symmetric

Sol As $(A - A^t)^t = A^t - (A^t)^t$

$(A - A^t)^t = A^t - A$

$\Rightarrow (A - A^t)^t = -(-A^t + A)$

$\Rightarrow (A-A^t)^t = -(A-A^t)$

$\Rightarrow A-A^t$ is skew symmetric matrix.

Q:5 Reduce each of the following matrices to the indicated forms.

(i) $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 3 & 4 & -5 \end{bmatrix}$ Echelon form.

Sol $R_2 - 2R_1$, $R_3 - 3R_1$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 6 \\ 0 & -5 & -2 \end{bmatrix}$$

Now $R_3 - R_2$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & -8 \end{bmatrix}$$

which is in Echelon form.

(ii) Sol $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$ Echelon form

$R_2 - 2R_1$, $R_3 - 3R_1$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 2 & 9 \end{bmatrix}$$

$R_3 - 2R_2$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$$ Ans

(iii) Sol $\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 2 \\ 4 & 1 & 7 \end{bmatrix}$ Reduced Row Echelon form

$R_1 \leftrightarrow R_2$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & 1 \\ 4 & 1 & 7 \end{bmatrix}$$

$R_2 - 2R_1$, $R_3 - 4R_1$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -3 \\ 0 & -3 & -1 \end{bmatrix}$$

$3R_2$ and $5R_3$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -15 & -9 \\ 0 & -15 & -5 \end{bmatrix}$$

Now $R_3 - R_2$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -15 & -9 \\ 0 & 0 & 4 \end{bmatrix}$$

Now $\frac{1}{4} R_3$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -15 & -9 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 + 9R_3$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -15 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-\frac{1}{15} R_2$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 - 2R_3$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 - R_2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is in R.R.E form

(iv) Sol $\begin{bmatrix} 0 & 2 & 3 \\ 3 & -4 & 1 \\ 1 & -1 & 2 \end{bmatrix}$ R.R.E form

$R_1 \longleftrightarrow R_3$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 \\ 3 & -4 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

$R_2 - 3R_3$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & -5 \\ 0 & 2 & 3 \end{bmatrix}$$

$R_3 + 2R_2$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & -5 \\ 0 & 0 & -7 \end{bmatrix}$$

$-\frac{1}{7} R_3$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 + 5R_3$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-R_2$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 - 2R_3$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 + R_2$

$$= \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is in R.R.E form

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Q:6 Find the inverses using ERO

(i) $\begin{bmatrix} 4 & -2 & 5 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$

Sol: let $A = \begin{bmatrix} 4 & -2 & 5 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$

First form the A/I matrix

$$A/I = \left[\begin{array}{ccc|ccc} 4 & -2 & 5 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \longleftrightarrow R_3$

$$\underset{R}{\sim} \left[\begin{array}{ccc|ccc} -1 & 2 & 3 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 1 & 0 & 0 \end{array}\right]$$

$-R_1$

$$\underset{R}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & 0 & -1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 1 & 0 & 0 \end{array}\right]$$

$R_2 - 2R_1\,,\ R_3 - 4R_1$

$$\underset{R}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & 0 & -1 \\ 0 & 5 & 6 & 0 & 1 & 2 \\ 0 & 6 & 17 & 1 & 0 & 4 \end{array}\right]$$

$R_2 - R_3$

$$\underset{R}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & 0 & -1 \\ 0 & -1 & -11 & -1 & 1 & -2 \\ 0 & 6 & 17 & 1 & 0 & 4 \end{array}\right]$$

$-R_2$

$$\underset{R}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & 0 & -1 \\ 0 & 1 & 11 & 1 & -1 & 2 \\ 0 & 6 & 17 & 1 & 0 & 4 \end{array}\right]$$

$R_3 - 6R_2$

$$\underset{R}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & 0 & -1 \\ 0 & 1 & 11 & 1 & -1 & 2 \\ 0 & 0 & -49 & -5 & 6 & -8 \end{array}\right]$$

$-\frac{1}{49}R_3$

$$\underset{R}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & 0 & -1 \\ 0 & 1 & 11 & 1 & -1 & 2 \\ 0 & 0 & 1 & \frac{5}{49} & \frac{-6}{49} & \frac{8}{49} \end{array}\right]$$

$R_2 - 11R_3$

$$\underset{R}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{-6}{49} & \frac{+17}{49} & \frac{10}{49} \\ 0 & 0 & 1 & 5/49 & -6/49 & 8/49 \end{array}\right]$$

$1 - \frac{55}{49} = \frac{49-55}{49}$

$-1 + \frac{66}{49} = \frac{-49+66}{49}$

$2 - \frac{88}{49} = \frac{98-88}{49}$

$-1 + \frac{24}{49}$

$R_1 + 3R_3$

$$\underset{R}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 15/49 & -18/49 & -25/49 \\ 0 & 1 & 0 & -6/49 & 17/49 & 10/49 \\ 0 & 0 & 1 & 5/49 & -6/49 & 8/49 \end{array}\right]$$

$R_1 + 2R_2$

$$\underset{R}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/49 & 16/49 & -5/49 \\ 0 & 1 & 0 & -6/49 & 17/49 & 10/49 \\ 0 & 0 & 1 & 5/49 & -6/49 & 8/49 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} 3/49 & 16/49 & -5/49 \\ -6/49 & 17/49 & 10/49 \\ 5/49 & -6/49 & 8/49 \end{bmatrix}$

or

$A^{-1} = \frac{1}{49}\begin{bmatrix} 3 & 16 & -5 \\ -6 & 17 & 10 \\ 5 & -6 & 8 \end{bmatrix}$ Ans

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(ii) $\begin{bmatrix} 3 & -1 & 6 \\ 1 & 3 & 4 \\ -1 & 5 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 3 & -1 & 6 \\ 1 & 3 & 4 \\ -1 & 5 & 1 \end{bmatrix}$

$A|I = \left[\begin{array}{ccc|ccc} 3 & -1 & 6 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ -1 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$

$R_1 \longleftrightarrow R_3$

$= \left[\begin{array}{ccc|ccc} -1 & 5 & 1 & 0 & 0 & 1 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 3 & -1 & 6 & 1 & 0 & 0 \end{array}\right]$

$-R_1$

$= \left[\begin{array}{ccc|ccc} 1 & -5 & -1 & 0 & 0 & -1 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 3 & -1 & 6 & 1 & 0 & 0 \end{array}\right]$

$R_2 - R_1 \ , \ R_3 - 3R_1$

$= \left[\begin{array}{ccc|ccc} 1 & -5 & -1 & 0 & 0 & -1 \\ 0 & 8 & 5 & 0 & 1 & 1 \\ 0 & 14 & 9 & 1 & 0 & 3 \end{array}\right]$

$\frac{1}{8} R_2$

$= \left[\begin{array}{ccc|ccc} 1 & -5 & -1 & 0 & 0 & -1 \\ 0 & 1 & 5/8 & 0 & 1/8 & 1/8 \\ 0 & 14 & 9 & 1 & 0 & 3 \end{array}\right]$

$R_3 - 14R_2$

$= \left[\begin{array}{ccc|ccc} 1 & -5 & -1 & 0 & 0 & -1 \\ 0 & 1 & 5/8 & 0 & 1/8 & 1/8 \\ 0 & 0 & 2/8 & 1 & -14/8 & 10/8 \end{array}\right]$

$9 - \frac{70}{8} \ , \ 3 - \frac{14}{8}$

$\frac{72-70}{8} \qquad \frac{24-14}{8}$

$\frac{8}{2} R_3$

$\frac{8}{2} \times \frac{-14}{8}$

$= \left[\begin{array}{ccc|ccc} 1 & -5 & -1 & 0 & 0 & -1 \\ 0 & 1 & 5/8 & 0 & 1/8 & 1/8 \\ 0 & 0 & 1 & 8/2 & -7 & 5 \end{array}\right]$

$= \left[\begin{array}{ccc|ccc} 1 & -5 & -1 & 0 & 0 & -1 \\ 0 & 1 & 5/8 & 0 & 1/8 & 1/8 \\ 0 & 0 & 1 & 4 & -7 & 5 \end{array}\right]$

$R_2 - \frac{5}{8} R_3$

$$= \left[\begin{array}{ccc|ccc} 1 & -5 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -5/2 & 9/2 & -3 \\ 0 & 0 & 1 & 4 & -7 & 5 \end{array}\right]$$

$\frac{1}{8} + \frac{35}{8} = \frac{36}{8}$

$\frac{5}{8} \times 4 = \frac{5}{2}$

$\frac{1}{8} - \frac{25}{8}$

$R_1 + R_3$

$$= \left[\begin{array}{ccc|ccc} 1 & -5 & 0 & 4 & -7 & 4 \\ 0 & 1 & 0 & -5/2 & 9/2 & -3 \\ 0 & 0 & 1 & 4 & -7 & 5 \end{array}\right]$$

$R_1 + 5R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -17/2 & 31/2 & -11 \\ 0 & 1 & 0 & -5/2 & 9/2 & -3 \\ 0 & 0 & 1 & 4 & -7 & 5 \end{array}\right]$$

$4 - \frac{25}{2} = \frac{-17}{2}$

$-7 + \frac{45}{2} = \frac{-14+45}{2}$

Hence $A^{-1} = \begin{bmatrix} -17/2 & 31/2 & -11 \\ -5/2 & 9/2 & -3 \\ 4 & -7 & 5 \end{bmatrix}$ Ans

Q.7: Find the rank

(i) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$

Sol: $R_2 - 2R_1$, $R_3 + R_1$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 5 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_3 - R_2$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 5 \\ 0 & 0 & -4 \end{bmatrix}$$

Since there are three non zero rows in Echelon form. Hence rank = 3.

(ii) $\begin{bmatrix} 3 & 1 & -4 \\ 0 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix}$

Sol

$R_1 \longleftrightarrow R_3$

$$= \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}$$

$R_3 - 3R_1$

$$= \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$

$R_3 - 2R_2$

$$= \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

we have 2 non-zero rows in Echelon form

Hence Rank = 2

Exercise # 2.5

Q:1 Solve by matrix method.

(i)
$$4x-3y+z=11$$
$$2x+y-4z=-1$$
$$x+2y-2z=1$$

Sol In matrix form, the system will be

$$\begin{bmatrix}4&-3&1\\2&1&-4\\1&2&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}11\\-1\\1\end{bmatrix}$$

let $\Downarrow A \quad \Downarrow X \quad \Downarrow B$

$$\Rightarrow AX=B$$
$$\Rightarrow X=A^{-1}B \longrightarrow (i)$$

To find A^{-1}, we have

$$|A| = 4\begin{vmatrix}1&-4\\2&-2\end{vmatrix}-(-3)\begin{vmatrix}2&-4\\1&-2\end{vmatrix}+1\begin{vmatrix}2&1\\1&2\end{vmatrix}$$
$$=4(-2+8)+3(-4+4)+1(4-1)$$
$$=4(6)+3(0)+1(3)$$
$$=24+0+3$$
$$|A|=27$$

Finding all the cofactors.

$$A=\begin{bmatrix}4&-3&1\\2&1&-4\\1&2&-2\end{bmatrix}$$

$$A_{11} = +\begin{vmatrix}1&-4\\2&-2\end{vmatrix} = +(-2+8)=6$$
$$A_{12} = -\begin{vmatrix}2&-4\\1&-2\end{vmatrix} = -(-4+4)=0$$
$$A_{13} = +\begin{vmatrix}2&1\\1&2\end{vmatrix} = +(4-1)=3$$
$$A_{21} = -\begin{vmatrix}-3&1\\2&-2\end{vmatrix} = -(6-2)=-4$$
$$A_{22} = +\begin{vmatrix}4&1\\1&-2\end{vmatrix} = +(-8-1)=-9$$
$$A_{23} = -\begin{vmatrix}4&-3\\1&2\end{vmatrix} = -(8+3)=-11$$
$$A_{31} = +\begin{vmatrix}-3&1\\1&-4\end{vmatrix} = +(12-1)=11$$
$$A_{32} = -\begin{vmatrix}4&1\\2&-4\end{vmatrix} = -(-16-2)=18$$
$$A_{33} = +\begin{vmatrix}4&-3\\2&1\end{vmatrix} = +(4+6)=10$$

Now
$$A\,adj=\begin{bmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{bmatrix}^t=\begin{bmatrix}6&0&3\\-4&-9&-11\\11&18&10\end{bmatrix}^t$$

$$\Rightarrow A\,adj=\begin{bmatrix}6&-4&11\\0&-9&18\\3&-11&10\end{bmatrix}$$

Then $A^{-1} = \frac{1}{|A|}[A\ adj]$

$$= \frac{1}{27}\begin{bmatrix} 6 & -4 & 11 \\ 0 & -9 & 18 \\ 3 & -11 & 10 \end{bmatrix}$$

Eqn (1) $\Rightarrow X = A^{-1}B$

$$\Rightarrow X = \frac{1}{27}\begin{bmatrix} 6 & -4 & 11 \\ 0 & -9 & 18 \\ 3 & -11 & 10 \end{bmatrix}\begin{bmatrix} 11 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{27}\begin{bmatrix} 66+4+11 \\ 0+9+18 \\ 33+11+10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 81 \\ 27 \\ 54 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Hence S. Set $= \{3, 1, 2\}$ Ans

(ii) $x + y + z = 1$
$x + y - 2z = 3$
$2x + y + z = 2$

Sol In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Let $\Downarrow A$, $\Downarrow X$, $\Downarrow B$

$$\Rightarrow AX = B \Rightarrow X = A^{-1}B \longrightarrow (i)$$

$$|A| = 1\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

$= 1(1+2) - 1(1+4) + 1(1-2)$

$= 1(3) - 1(5) + 1(-1)$

$= 3 - 5 - 1$

$|A| = -3$

Now cofactors

$A_{11} = +\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = +(1+2) = 3$

$A_{12} = -\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = -(1+4) = -5$

$A_{13} = +\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = +(1-2) = -1$

$A_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -(1-1) = 0$

$A_{22} = +\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = +(1-2) = -1$

$A_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(1-2) = +1$

$A_{31} = +\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3$

$A_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = +3$

$A_{33} = +\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

Now $A\ adj = \begin{bmatrix} 3 & -5 & -1 \\ 0 & -1 & 1 \\ -3 & 3 & 0 \end{bmatrix}^t \Rightarrow A\ adj = \begin{bmatrix} 3 & 0 & -3 \\ -5 & -1 & 3 \\ -1 & 1 & 0 \end{bmatrix}$

Now $A^{-1} = \frac{1}{|A|}[A\ adj]$

$$= \frac{1}{-3}\begin{bmatrix} 3 & 0 & -3 \\ -5 & -1 & 3 \\ -1 & 1 & 0 \end{bmatrix}$$

Eqn (i) $\Rightarrow$ $X = A^{-1}B$

$$\Rightarrow X = -\frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ -5 & -1 & 3 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\Rightarrow X = \frac{-1}{3}\begin{bmatrix} 3+0-6 \\ -5-3+6 \\ -1+3+0 \end{bmatrix}$$

$$\Rightarrow X = -\frac{1}{3}\begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}\times(-3) \\ -\frac{1}{3}\times -2 \\ -\frac{1}{3}\times 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2/3 \\ -2/3 \end{bmatrix}$$

S. Set $= \{1, 2/3, -2/3\}$ Ans

Q:2 Solve by Gauss Elimination (Echelon) and Gauss-Jorden (Reduced-Row Echelon form) method. CH-02 P-21

(i)
$$x - y + 4z = 4$$
$$2x + 2y - z = 2$$
$$3x - 2y + 3z = -3$$

Gauss Elimination (Echelon form) Method:

Here the coefficient matrix $A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 2 & -1 \\ 3 & -2 & 3 \end{bmatrix}$

& $B = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$

The Augmented matrix (A/B) will be

$$A/B = \left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 2 & 2 & -1 & 2 \\ 3 & -2 & 3 & -3 \end{array}\right]$$

$R_2 - 2R_1, \quad R_3 - 3R_1$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 0 & 4 & -9 & -6 \\ 0 & 1 & -9 & -15 \end{array}\right]$$

$4R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 0 & 4 & -9 & -6 \\ 0 & 0 & -27 & -54 \end{array}\right]$$

Now the system is in Echelon form

From R_3, we make the eqn

$$0x + 0y - 27z = -54$$
$$\Rightarrow -27z = -54$$
$$\Rightarrow \boxed{z = 2}$$

From R_2 : $0x + 4y - 9z = -6$
$$\Rightarrow 4y - 9(2) = -6$$
$$\Rightarrow 4y - 18 = -6$$
$$\Rightarrow 4y = 12 \Rightarrow \boxed{y = 3}$$

From R_1 : $1x - 1y + 4z = 4$
$$\Rightarrow x - 3 + 4(2) = 4$$
$$\Rightarrow x + 5 = 4 \Rightarrow \boxed{x = -1}$$

Hence S. Set $= \{-1, 3, 2\}$ Ans

<u>By Gauss – Jordan Method (Reduced Row Echelon form)</u>

We have the system $\left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 0 & 4 & -9 & -6 \\ 0 & 0 & -27 & -54 \end{array}\right]$

$-\frac{1}{27} R_3$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 0 & 4 & -9 & -6 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_2 + 9R_3$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 0 & 4 & 0 & 12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Now $\frac{1}{4} R_2$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 4 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_1 - 4R_3$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_1 + R_2$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Hence S. Set $= \{-1, 3, 2\}$ Ans

(ii)
$$2x + 4y - z = 0$$
$$x - 2y - 2z = 2$$
$$-5x - 8y + 3z = -2$$

<u>Sol</u> $A = \begin{bmatrix} 2 & 4 & -1 \\ 1 & -2 & -2 \\ -5 & -8 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

<u>By Gauss Elimination method</u>

Augmented matrix $A|B = \left[\begin{array}{ccc|c} 2 & 4 & -1 & 0 \\ 1 & -2 & -2 & 2 \\ -5 & -8 & 3 & -2 \end{array}\right]$

$$R_1 \longleftrightarrow R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 2 & 4 & -1 & 0 \\ -5 & -8 & 3 & -2 \end{array}\right]$$

$R_2 - 2R_1, \quad R_3 + 5R_1$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 8 & 3 & -4 \\ 0 & -18 & -7 & 8 \end{array}\right]$$

$\frac{1}{4} R_2$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 2 & 3/4 & -1 \\ 0 & -18 & -7 & 8 \end{array}\right]$$

$R_3 + 9R_2$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 2 & 3/4 & -1 \\ 0 & 0 & -1/4 & -1 \end{array}\right]$$

$$-7 + \frac{27}{4} = \frac{-28+27}{4}$$

$-4 R_3$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 2 & 3/4 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

It is Echelon form

From R_3 $\quad 0x + 0y + 1z = 4$

$$\Rightarrow \boxed{z = 4}$$

From R_2

$$0x + 2y + \frac{3}{4}z = -1$$

$$\Rightarrow 2y + \frac{3}{4}(4) = -1$$

$$\Rightarrow 2y + 3 = -1$$

$$\Rightarrow 2y = -4 \Rightarrow \boxed{y = -2}$$

From R_1

$$1x - 2y - 2z = 2$$

$$\Rightarrow x - 2(-2) - 2(4) = 2$$

$$\Rightarrow x + 4 - 8 = 2$$

$$\Rightarrow x - 4 = 2 \Rightarrow \boxed{x = 6}$$

S.Set $= \{6, -2, 4\}$ Ans

By Gauss - Jordan Method:

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 2 & 3/4 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$4R_2$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 8 & 3 & -4 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$R_2 - 3R_3$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 8 & 0 & -16 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Quote: The happiest of people don't necessarily have the best of everything. They just make the most of everything that comes along their way.

$\frac{1}{8}R_2$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$R_1 + 2R_3$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 0 & 10 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$R_1 + 2R_2$

99 $$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$ Hence S.set $= \{6, -2, 4\}$

Q.3 Solve the systems by Cramer's Rule

(i)
$$x - 2y + 0z = -4$$
$$3x + y + 0z = -5$$
$$2x + 0y + z = -1$$

Sol The coefficient matrix A is

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} -4 \\ -5 \\ -1 \end{bmatrix}$$

Expand by C_3

$$|A| = +0 - 0 + 1\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}$$

$$\Rightarrow |A| = +1(1+6) \Rightarrow \boxed{|A| = 7}$$

To find x:

Replace C_1 of A by B and then divide its determinant by $|A|$

i.e
$$x = \frac{\begin{vmatrix} -4 & -2 & 0 \\ -5 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}}{|A|} = \frac{0 - 0 + 1\begin{vmatrix} -4 & -2 \\ -5 & 1 \end{vmatrix}}{7}$$

$$= \frac{1(-4-10)}{7} = \frac{-14}{7} = -2$$

To find y: Replace C_2 of A by B and then divide its ~~det~~ by $|A|$.

i.e
$$y = \frac{\begin{vmatrix} 1 & -4 & 0 \\ 3 & -5 & 0 \\ 2 & -1 & 1 \end{vmatrix}}{|A|} = \frac{1\begin{vmatrix} 1 & -4 \\ 3 & -5 \end{vmatrix}}{7} = \frac{-5+12}{7} = \frac{7}{7} = 1$$

To find z: Replace C_3 of A by B and then divide its det by $|A|$.

$$z = \frac{\begin{vmatrix} 1 & -2 & -4 \\ 3 & 1 & -5 \\ 2 & 0 & -1 \end{vmatrix}}{|A|} = \frac{1\begin{vmatrix} 1 & -5 \\ 0 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 3 & -5 \\ 2 & -1 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix}}{7}$$

$$= \frac{1(-1-0) + 2(-3+10) - 4(0-2)}{7}$$

$$= \frac{-1+14+8}{7} = \frac{21}{7} = 3$$

Hence S.set $= \{-2, 1, 3\}$ Ans

(ii)
$$x - y + 2z = 10$$
$$2x + y - 2z = -4$$
$$3x + y + z = 7$$

Sol: $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -2 \\ 3 & 1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 10 \\ -4 \\ 7 \end{bmatrix}$

$$\Rightarrow |A| = 1\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$$
$$= 1(1+2) + 1(2+6) + 2(2-3)$$
$$= 1(3) + 1(8) + 2(-1)$$
$$= 3 + 8 - 2 \quad \Rightarrow |A| = 9$$

To find x: Replace C_1 of A by B and divide its det by $|A|$

i.e
$$x = \frac{\begin{vmatrix} 10 & -1 & 2 \\ -4 & 1 & -2 \\ 7 & 1 & 1 \end{vmatrix}}{|A|} = \frac{10\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -4 & -2 \\ 7 & 1 \end{vmatrix} + 2\begin{vmatrix} -4 & 1 \\ 7 & 1 \end{vmatrix}}{9}$$
$$= \frac{10(1+2) + 1(-4+14) + 2(-4-7)}{9}$$
$$= \frac{30 + 10 - 22}{9} = \frac{18}{9} = 2$$

Similarly
$$y = \frac{\begin{vmatrix} 1 & 10 & 2 \\ 2 & -4 & -2 \\ 3 & 7 & 1 \end{vmatrix}}{|A|} = \frac{+1\begin{vmatrix} -4 & -2 \\ 7 & 1 \end{vmatrix} - 10\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & -4 \\ 3 & 7 \end{vmatrix}}{9}$$

$$\Rightarrow y = \frac{1(-4+14) - 10(2+6) + 2(14+12)}{9}$$
$$y = \frac{10 - 80 + 52}{9} = \frac{-18}{9} = -2$$

Finally,
$$z = \frac{\begin{vmatrix} 1 & -1 & 10 \\ 2 & 1 & -4 \\ 3 & 1 & 7 \end{vmatrix}}{|A|} = \frac{1\begin{vmatrix} 1 & -4 \\ 1 & 7 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -4 \\ 3 & 7 \end{vmatrix} + 10\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}}{9}$$
$$\Rightarrow z = \frac{1(7+4) + 1(14+12) + 10(2-3)}{9}$$
$$\Rightarrow z = \frac{11 + 26 - 10}{9} = \frac{27}{9} = 3$$

Hence S.set $= \{2, -2, 3\}$ Ans

Q:4 Solve the system of homogeneous eqns.

(i)
$$x_1 - x_2 + x_3 = 0$$
$$x_1 + 2x_2 - x_3 = 0$$
$$2x_1 + x_2 + 3x_3 = 0$$

Sol: The coefficient matrix A is $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Then $|A| = \begin{vmatrix}1 & -1 & 1\\ 1 & 2 & -1\\ 2 & 1 & 3\end{vmatrix}$

Expand by R_1

$\Rightarrow |A| = 1\begin{vmatrix}2 & -1\\ 1 & 3\end{vmatrix} - (-1)\begin{vmatrix}1 & -1\\ 2 & 3\end{vmatrix} + 1\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix}$

$\Rightarrow |A| = 1(6+1) + 1(3+2) + 1(1-4)$

$\Rightarrow |A| = 1(7) + 1(5) + 1(-3)$

$\Rightarrow |A| = 7 + 5 - 3$

$\Rightarrow |A| = 9$

Since $|A| \neq 0 \Rightarrow$ Trivial solution only $\{0,0,0\}$

Hence $S.Set = \{0,0,0\}$ Ans

(ii)

$$x_1 + x_2 + 2x_3 = 0$$
$$-2x_1 + x_2 - x_3 = 0$$
$$-x_1 + 5x_2 + 4x_3 = 0$$

Sol: $A = \begin{bmatrix}1 & 1 & 2\\ -2 & 1 & -1\\ -1 & 5 & 4\end{bmatrix}$

$|A| = 1\begin{vmatrix}1 & -1\\ 5 & 4\end{vmatrix} - 1\begin{vmatrix}-2 & -1\\ -1 & 4\end{vmatrix} + (2)\begin{vmatrix}-2 & 1\\ -1 & 5\end{vmatrix}$

$= 1(4+5) - 1(-8-1) + 2(-10+1)$

$\Rightarrow |A| = 1(9) - 1(-9) + 2(-9)$

$\Rightarrow |A| = 9 + 9 - 18 = 0$

Since $|A| = 0 \Rightarrow$ Non-Trivial solution is also possible.

Now $A|B = \left[\begin{array}{ccc|c}1 & 1 & 2 & 0\\ -2 & 1 & -1 & 0\\ -1 & 5 & 4 & 0\end{array}\right]$

$R_2 + 2R_1, \quad R_3 + R_1$

$= \left[\begin{array}{ccc|c}1 & 1 & 2 & 0\\ 0 & 3 & 3 & 0\\ 0 & 6 & 6 & 0\end{array}\right]$

$R_3 - 2R_2$

$= \left[\begin{array}{ccc|c}1 & 1 & 2 & 0\\ 0 & 3 & 3 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$

Let $x_3 = t$

Then from R_2, we have

$0x_1 + 3x_2 + 3x_3 = 0$

$\Rightarrow 3x_2 + 3t = 0$

$\Rightarrow 3x_2 = -3t$

$\Rightarrow \boxed{x_2 = -t}$

From R_1

$1x_1 + 1x_2 + 2x_3 = 0$

$x_1 - t + 2t = 0$

$x_1 + t = 0$

$\Rightarrow x_1 = -t$

Hence $S.Set = \{-t, -t, t\}$

Q:5 For what value of λ, the system has non-Trivial solution. Solve the system.

$$x_1 + 5x_2 + 3x_3 = 0$$
$$5x_1 + x_2 - \lambda x_3 = 0$$
$$x_1 + 2x_2 + \lambda x_3 = 0$$

Sol

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 5 & 1 & -\lambda \\ 1 & 2 & \lambda \end{bmatrix}$$

For non Trivial solution

$$|A| = 0$$

$$\Rightarrow 1\begin{vmatrix} 1 & -\lambda \\ 2 & \lambda \end{vmatrix} - 5\begin{vmatrix} 5 & -\lambda \\ 1 & \lambda \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow 1(\lambda + 2\lambda) - 5(5\lambda + \lambda) + 3(10-1) = 0$$

$$\Rightarrow 1(3\lambda) - 5(6\lambda) + 3(9) = 0$$

$$\Rightarrow 3\lambda - 30\lambda + 27 = 0$$

$$\Rightarrow -27\lambda + 27 = 0$$

$$\Rightarrow 27\lambda = 27 \Rightarrow \boxed{\lambda = 1} \text{ Ans}$$

Then $A = \begin{bmatrix} 1 & 5 & 3 \\ 5 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

$$A|B = \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 5 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right]$$

$R_2 - 5R_1$; $R_3 - R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & -24 & -18 & 0 \\ 0 & -3 & -2 & 0 \end{array}\right]$$

$-\frac{1}{4} R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & +6 & 4 & 0 \\ 0 & -3 & -2 & 0 \end{array}\right]$$

$R_3 + 2R_2$

$$= \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 6 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = t$

From R_2 : $0x_1 + 6x_2 + 4x_3 = 0$

$$6x_2 + 4t = 0$$
$$6x_2 = -4t$$

$$x_2 = -\frac{2}{3}t$$

From R_1

$$1x_1 + 5x_2 + 3x_3 = 0$$
$$x_1 + 5\left(\frac{-2}{3}t\right) + 3t = 0$$
$$x_1 - \frac{10}{3}t + 3t = 0$$
$$x_1 - \frac{t}{3} = 0 \Rightarrow x_1 = \frac{t}{3}$$

Hence S. Set $= \left\{\frac{t}{3}, \frac{-2}{3}t, t\right\}$

Hurrah! We have finished chapter #2